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The Journal of Experimental Education

A periodical report of scientific investigations relating to child development, curriculum, learning, teaching, supervision, measurements, statistics, and experimental techniques.

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Journal of Experimental Education

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THE ASSOCIATION OF MATHEMATICAL ACHIEVEMENT WITH CERTAIN FACTORS RESIDENT IN THE TEACHER, IN THE TEACHING, IN THE PUPIL, AND IN THE SCHOOL*

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In recent years the content, instruction, and product of the mathematics program in the secondary school have been subjected to frequent criticism. Studies seeking to establish factual information concerning the issues have been fragmentary and unconvincing. Meanwhile enrollments in mathematics have continued to decrease in a technological society in which the need for mathematical proficiency has been expanding. There is evident need for carefully conducted and comprehensive research in this area. Accurate description and analysis of the present situation is essential as the first step toward eventual solution of the existing problems. This study was undertaken with the intent of contributing toward the satisfaction of that need.

Statement of the Problem

The problem was to determine the status of mathematics instruction in Minnesota's public secondary schools and to investigate the relation of mathematical achievement to certain factors resident in the teaching, in the teacher, in the pupil, and in the school.

The scope of the study included: (a) investigation of the status of instruction in three subjects, namely, general mathematics, elementary algebra, and plane geometry; and (b) investigation of the association of selected factors with mathematical achievement in two subjects, namely, elementary algebra and plane geometry.

Specifically, the primary purpose of the study was to investigate: 1. The relation of mathematical achievement with the training and experience of the teacher; 2. the relation of mathematical achievement with the sex, social background, and educational plans of the pupil; 3. the relation of mathematical achievement with the class-size and the size and organization of the school; and 4. the relation of mathematical achievement with twelve selected factors resident in the methods and materials of instruction.

A subsidiary function of the study was to discover what practices were followed in the teaching of elementary algebra, general mathematics, and plane geometry in Minnesota schools; and to determine the nature of the professional training, teaching experience, teaching load, and professionalism of the teachers of those subjects. The status of these items was investigated so that selected factors could be evaluated in terms of pupil achievement.

Design of the Study

The study was conducted during the 1947-48 school year. The methodology was that of the sample survey. The employed sample of one hundred schools was selected by the method of stratified-proportionate sampling from the population of Minnesota's 522 public secondary schools classified with respect to enrollment size and organizational type.

* Summary of an unpublished doctoral dissertation of the same title, University of Minnesota, 1950. The second in a series of studies devoted to the analysis of achievement in specific subject matter areas. See K. E. Anderson, "A Frontal Attack on the Basic Study in Evaluation," Journal of Experimental Education, XVI (March 1950), pp. 163-174.

From the sample, three objective measures of pupil characteristics were secured. Mathematical achievement was measured at the beginning and at the end of the school year by means of identical examinations constructed by the investigator. These examinations were based upon authoritative objectives[1,2] and placed approximately equivalent emphasis on the three areas of (a) knowledge of mathematical concepts and principles, (b) mastery of mathematical skills, and (c) application of mathematical knowledge and skills to the solution of practical problems.[3] The items were carefully selected after critical examination by a group of experienced teachers and statistical validation on the basis of a pilot study conducted in six schools at the conclusion of the 1946-47 school year. The reliability coefficients of the tests ranged from .89 to .94 as computed by Hoyt's[4] method on samples of the pupils measured. The tests were judged to be valid and usable by a committee of experts and by an independent sample of teachers in the field. The Otis Quick-Scoring Mental Ability Test was the third objective pupil-measure employed.

In addition to the objective measures, a questionnaire concerning the age, sex, educational plans, and social background of the pupils, as well as a ninety-item schedule descriptive of the teachers' experience, training, and instructional practices were used in the data-gathering phase of the study. The schedule was organized so as to ascribe quantitative limits to qualitative categories such as "seldom," "frequently," and "regularly." In the ensuing statistical analysis, the achievement of pupils taught by teachers who indicated regular use of a specific practice was compared with the achievement of pupils taught by teachers who indicated that they seldom used the practice under study.

The statistical technique employed in the comparisons was that of analysis of variance and covariance. The criterion variable was mathematical achievement as measured by the Minnesota State Board Examination (constructed by the investigator and identical with the initial tests). Any inequalities with respect to Otis mental score and initial mathematics achievement among the groups compared were controlled by the analysis of covariance. Furthermore, by a process of classification and random selection within strata, it was possible to control such additional factors as school-size, school-organization, teacher-experience, and teacher mathematical-training, when other factors were under comparison. The appropriate null hypothesis was tested in thirty-four separate analyses.

The Minnesota State Department of Education, through its directors of curriculum and testing, cooperated fully in the field work of the study. All of the original contacts with participating schools were through official channels, and much of the followup and testing was accomplished under State Department authority. This cooperation was the keystone to efficient execution of the project.

Procedures and Techniques

Selection of the Sample. —The sample of one hundred schools was selected from the population by the process of stratified random sampling. This technique of sampling follows the general procedure of simple random sampling, but takes a preliminary step. The population of size N is first divided on the basis of certain criteria into subpopulations of sizes N_1, N_2, —N_k. These subpopulations are called strata. When the strata have been determined, a simple random sample is then taken from each stratum independently. The sample sizes within the strata are then n_1, n_2, —n_k: optimum allocation. Stratification increases the accuracy of the sample by dividing a heterogeneous population into homogeneous strata. For this reason the method was employed in this study.

To be effective, it was necessary to select a stratification based upon some known population characteristic which might reasonably be presumed to be systematically associated with the criterion variable (mathematical achievement). From the limited available prior-knowledge concerning the population, school enrollment and administrative organization were selected as the most feasible bases for stratification. The enrollment strata employed were seven, namely, under 50 pupils, 50 to 99 pupils, 100 to 199 pu-

1. The Place of Mathematics in Secondary Education, Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics. Final Report of the Commission (1940). Pp. 62-68.

2. Guidance Pamphlet in Mathematics for High School Students, Post-War Planning Commission of the National Council of Teachers of Mathematics. Final Report of the Commission (1947).

3. The Place of Mathematics in Secondary Education, op. cit., p. 61.

4. Cyril J. Hoyt, "Test Reliability Estimated by Analysis of Variance," Psychometrika, VI, pp. 153-160.

pils, 200 to 299 pupils, 300 to 499 pupils, 500 to 999 pupils, and over 1000 pupils. Six administrative classifications were employed: four-year schools, six-year schools, junior-senior high schools, junior high schools, senior high schools, and high school departments.

Prior to the drawing of the sample, each of the 522 secondary schools listed in the Minnesota Educational Directory[5] was assigned a number and categorized. The proportion of schools in each of the forty-two categories was determined. A sample of one hundred schools was then drawn in such a manner that each stratum received proportionate representation. For example, twenty-one of Minnesota's secondary schools were classified in the category indicating four-year schools of less than fifty enrollment. Then, since twenty-one is 4 percent of 522, four schools were drawn by simple random processes from the twenty-one. Thus, a 4 percent representation of that category was attained in the sample. Tippett's table of random numbers was employed in the drawing.[6] In similar manner, the proper representation for each of the remaining categories was determined and selected.

Testing the Representativeness of the Completed Sample. —In any study which requires cooperation of large numbers of individuals over a full academic year, it is difficult, if not impossible, to secure 100 percent response. In this study followup by letter, telephone, or personal conference was employed in an effort to reduce the non-respondents to negligible proportions. Nevertheless, from the drawn sample which included 139 elementary algebra classes and 114 classes in plane geometry, complete returns were received on 102 classes in elementary algebra and 94 classes in plane geometry. This was a 77 percent return by classes. In the classes which completed the study, 3919 pupils were enrolled. However, absences on designated test days, program changes, and transfers to other schools resulted in incomplete returns from 355 elementary algebra pupils and 279 pupils of plane geometry. This was a further loss of 14 percent. Therefore, since the validity of the findings in a survey is dependent upon the representativeness of the completed sample, it was necessary to study several questions:

1. Were the schools which submitted complete data significantly different with respect to their administrative type and enrollment size from the schools for which the data were incomplete?

2. Were the teachers who taught the classes which completed the study significantly different in training and experience from the teachers of classes which failed to finish the project?

3. Were the classes which completed the study significantly different in enrollment size from the classes which failed to finish the study?

4. Were the initial test mean-scores and mental test mean-scores of classes which completed the study significantly different from the same measures of classes for which data were incomplete?

5. Were the mean initial-test scores and mean mental-test scores of pupils who completed the project significantly different from those measures of their classmates for whom the returns were incomplete?

Employing data on file at the State Department of Education to supplement the partial returns of those who failed to complete the study, the null hypothesis was tested by chi-square technique for points one through three, above, and by the Behrens-Fisher d test for points four and five.

In these fourteen comparisons of respondents and non-respondents, holding the .01 level of significance, it was concluded that:

1. The respondent schools did not differ significantly from the non-respondent schools in respect to organizational type. School enrollment size was proportionately represented in the algebra sample, but in the geometry sample the representation of schools enrolling over 500 pupils was 14 percent greater than optimum.

2. The teachers who taught the classes which completed the study did not differ significantly from the teachers of classes which failed to complete all phases of the project. The teacher characteristics under comparison were: (a) type of institution from which they were graduated, and (b) number of years of teaching experience.

3. The geometry classes which completed the study did not differ significantly in enrollment size from the non-respondent classes. However, in the algebra sample, classes enrolling more than thirty pupils received a disproportionately small representation. The ratio of classes enrolling less than twenty pupils, from twenty to thirty pupils, and more than thirty pupils, respectively, was 34:57:11 in the completed sample and 9:15:13 in the non-respondent group.

4. The pupils in the classes that completed the study were not significantly different from the pupils of non-respondent classes with respect to initial mathematics score and mental score. This conclusion was tentative, since test data on only a part of the non-respondents were available.

5. Within the respondent classes, individual

5. Minnesota Educational Directory, 1946-1947, Minnesota Department of Education (St. Paul: Minnesota Department of Education, 1947).

6. L. H. C. Tippett, Tables of Random Sampling Numbers, Tracts for Computers, No. 15 (London: Cambridge University Press, 1927).

pupils who failed to take all the tests were significantly inferior in mean mental-score and in mean initial-test score to their classmates for whom the test data were complete.

Thus, the findings, in general, indicated that the samples used in the analyses were representative of the population with respect to the factors studied. Moreover, in the ensuing analyses of achievement, possibility of bias attributable to disproportionate representation in the completed sample was reduced or eliminated by employment of randomly selected equivalent representation of pertinent categories in each group of subsamples under comparison. It was assumed that the bias, if it existed, would then be approximately constant from subgroup to subgroup so that while estimates of parameter values might be somewhat affected, the possibility of non-response bias, even though remote, must be given consideration in the interpretation of the specific findings of the study.

Selection of Subsamples. —The size of the completed sample prohibited employment of all the data in each analysis of factors presumed to be associated with mathematical achievement. In the majority of comparisons, subsamples of from eight to twenty classes were selected to represent the larger group possessing the characteris_ics under study. This constituted a reduction of approximately 50 percent in the amount of data studied. In order to minimize the possibility of bias, the subsamples were selected by the process of stratified-proportionate sampling. Through this procedure, variables in addition to those held constant by covariance techniques could be approximately controlled by equal representation among the subsamples. An illustration of this procedure follows:

Among the 139 classes in the completed algebra sample, there were twenty-five classes in which differentiated assignments were regularly used and forty-five classes in which identical problems for all pupils were regularly assigned. These seventy classes, classified into homogeneous subgroups with respect to school-size and teacher-experience, were then placed in their proper cell in Table I. These two variables had previously been shown to be significantly associated with algebra achievement. In the table it can be observed that classes in cells A, B, C, and D employ differentiated assignments; whereas, classes in cells A', B', C', and D' employ identical assignments. The two categories of enrollment size are: "R" (schools enrolling from 100 to 500 pupils) and "S" (schools enrolling either less than 100 pupils or more than 500 pupils). Earlier analyses had shown that the schools in category "R" did not differ as to class mean-achievement among themselves, but did differ in class mean-achievement from schools in category "S". The teacher-experience categories, Group I (less than eight years of teaching) and Group II (more than eight years of teaching), had also been shown to differ in mean algebra achievement.

Thus, in the table, cells A and A', B and B', C and C', and D and D', respectively, contain comparable classes with respect to school-size and teacher experience. The subsample of thirty-two classes was then randomly drawn from within cells so that the representation within each pair of cells was equal. The classes drawn are underlined in Table I.

Analysis of subsamples. —In this study, thirty-four comparisons of mathematical achievement of subsamples were made by analysis of variance and covariance techniques. In each comparison, the null hypothesis was under test. One complete analysis is summarized here to illustrate the general procedure.

In this analysis the following major hypothesis H_4 was tested: "When the teacher-factors of amount of experience and amount of college mathematical training were controlled by matching, and the pupil-factors of mental score and initial algebra achievement were controlled by co-variance technique, there were no significant differences in mean algebra achievement in classes taught by teachers who earned their undergraduate degrees at private colleges, teachers' colleges, or state universities, respectively."

From the teacher schedule, it was determined that of the sixty-three teacher of algebra in the study, twenty-three had received their professional education at teacher's colleges, twenty-one were trained at private colleges, and fifteen completed the teacher-education curricula at state universities. (Of the remaining four teachers, three had not earned a degree and one failed to respond to the question.) By a procedure similar to that illustrated in Table I, a sample of twenty-four classes was drawn. The eight classes in each subsample were drawn in such a manner that four were taught by teachers who had studied more than two years of college mathematics, and four were taught by teachers whose college mathematics training was limited to two years or less. Likewise, the eight classes in each subsample satisfied the imposed teacher-experience restriction. Four classes in each were taught by teachers who had more than eight years of experience, and the remaining four were taught by teachers of eight years' experience or less.

Whenever analysis of variance and covariance technique is employed, the data to be analyzed must satisfy certain basic assumptions. They are:

1. The variables under consideration must be approximately normally distributed in the population.

2. The classes to be grouped together or

TABLE I

CLASSIFICATION OF ALGEBRA CLASSES FOR THE PURPOSE OF STRATIFIED SAMPLING*

School Enroll.	Classes (by Number) Using Differentiated Assignments		Classes (by Number) Using Identical Assignments	
	I[a]	II[b]	I[a]	II[b]
(R)	(A)	(B)	(A')	(B')
100-499	21 26 27 50 51 52 53 57	16 30 31 33 37 41 42 59	14 15 20 25 32 40 44 45 47 61 24 46	22 23 36 39
(S)	(C)	(D)	(C')	(D')
<100 or >500	2 12	7 8 81 82 83 88 89	1 4 6 10 18 71 72 73 84 85 97 98 101 100 99	9 13 74 75 76 77 78 79 80 86 87 90 91 92

*Classes drawn in the subsample are underlined.

[a]Classes of teachers having less than eight years of experience.

[b]Classes of teachers having eight or more years of experience.

Table I illustrates the organization of data when two variables were controlled by equal representation in the subgroups under comparison. As the analysis of the various factors progressed, additional variables were employed in the classification of subgroups.

TABLE II

WELCH-NAYER TEST FOR HOMOGENEITY OF VARIANCE - CLASSES OF TEACHERS WHO EARNED THEIR UNDERGRADUATE DEGREES AT STATE UNIVERSITIES
(Group C)

Class	n_S	$1/n_S-1$	$n_S \log n_S$	θ_S	$n_S \log \theta_S$
1	7	.1667	5.9156860	952.8574	20.8531946
26	21	.0500	27.7666053	6368.9524	79.8854280
39	39	.0263	62.0515194	11982.9231	159.0639492
79	21	.0500	27.7666053	4601.2381	76.9203687
83	19	.0556	24.2963184	3908.7368	68.2486916
90	24	.0435	33.1250688	3596.6250	85.3414824
Σ	131	.3921	180.9218032	31411.3325	490.3131145
Log Σ	2.1172713			4.4970864	

	Harmonic Mean	No. of Groups	Log L_1	L_1	P	Null Hypothesis
Reduced Pool	15	6	9.9819506-10	.959	.05	Accept

"pooled" must be homogeneous with respect to the variances of the dependent variable and, also, with respect to the means of the dependent variable.

3. The "within groups" regression of the dependent variable upon the independent variables in the pooled groups under study must be homogeneous.

The first basic assumption was tested by application of chi-square tests to the distributions of initial mathematics scores, mental-test scores and final algebra scores. These tests indicated that the distributions deviated somewhat from true normality. However, examination of the several histograms together with consideration of the computed coefficients of skewness and kurtosis revealed that the distributions were unimodal and departed only slightly from the normal model. Since Fisher[7] had shown that slight departure from normality of distribution among the variables causes only moderate reduction in efficiency of the analysis, transformations of the data were not considered necessary.[8]

The second basic assumption was tested by first applying the Welch-Nayer L_1 test to the hypothesis H_1, "the eight classes in each subsample had been drawn from normal populations having a common standard deviation;" and then using Snedecor's F test on the hypothesis H_2, "the classes in each subsample had been drawn from normal populations having a common mean." (Since homogeneity of variance is an assumption basic to the F test, it was necessary to test the variance within subsamples prior to testing their differences in means.)

For Group C, classes taught by teachers who had earned their undergraduate degree at a state university, an L_1 value of .905 was obtained. Entering Nayer's tables with k (the number of independently drawn classes in the subsample) equal to 8, and with fifteen degrees of freedom (the harmonic mean of one less than the number of pupils in each of the k classes), the tabled value for L_1 was less than .905 at the .05 level. The hypothesis H_1 was then accepted. The next step involved the F test of H_2.

The data for the eight classes in Group C produced a "between groups mean square" of 1103.4656 and a "within groups mean square" of 223.4406. The resulting variance ratio of 4.939 was greater than Snedecor's F value at the .05 level. The table was entered with n_1 (the degrees of freedom of the greater mean square) equal to 7 and n_2 (the degrees of freedom of the lesser mean square) equal to 156. Therefore, hypothesis H_2 was rejected at the .05 level. The subsample as drawn did not satisfy the assumption of homogeneity of means among its eight elements. In order to proceed toward the analysis of covariance, it was necessary to remove from the analysis those classes which prevented the acceptance of H_2.[9] Inspection revealed that the means of two classes deviated atypically from the Group C mean. Removal of the two classes reduced Group C to six classes which when retested by the L_1 and F tests permitted pooling of the group (Table II and Table III).

The restrictions of pooling reduced Group B (classes of teachers who had earned their undergraduate degrees at private colleges) to four schools. Group A, the subsample of classes taught by teachers who had earned their undergraduate degrees at teachers' colleges, was successfully pooled after having been reduced to five classes. The three subsamples were then ready for pooled analysis.[10]

The next hypothesis, "the three pooled subsamples were drawn from populations having a common variance," was tested by applying the Welch-Nayer L_1 test to the sums of squares with-

7. R. A. Fisher, "On the Mathematical Foundations of Theoretical Statistics," Philosophical Transactions of the Royal Society of London, A, 222 (1922), pp. 309-368.

8. W. G. Cochran, "Some Consequences When the Assumptions for the Analysis of Variance Are Not Satisfied," Biometrics, (March 1947).

9. Almost without exception in this study, the subsamples as originally drawn, failed to satisfy the assumption of homogeneity of means among the classes to be pooled. This extreme heterogeneity of class-means in the subsamples posed a difficult problem. What procedure would achieve the required homogeneity and also avoid introduction of bias into the reduced subsamples? Transformation of scores was not practical considering the large amount of data involved. Likewise, selection of classes to be eliminated by random means was not feasible since it might have seriously changed the means of reduced subsamples. The problem was solved by removing atypical classes from both extremes of the distribution of class-means in such a manner that the original group-mean was practically undisturbed. By a process of trial and error it was possible to find the minimum number of eliminations which would achieve homogeneity of means and yet not change the original-group mean by more than 1 percent.

10. Data removed from the reported analyses were placed on file, available for possible later separate analysis and reports.

TABLE III

ANALYSIS OF VARIANCE OF THE END SCORES TO DETERMINE THE HOMOGENEITY OF MEANS FOR PURPOSES OF POOLING (Group C)

Source of Variation	d. f.	Sum of Squares	Mean Square	F	Prob-ability	Null Hypothesis
Within Groups	125	31411.3325	251.2907			
Between Groups	5	2267.4614	453.4922	1.805	.05	Accept
Total	130	33678.7939				

TABLE IV

DATA FOR ANALYSIS OF VARIANCE AND COVARIANCE - CLASSES GROUPED ACCORDING TO TEACHER-EDUCATION INSTITUTIONS

Group	N	Σw^2	Σwy	Σy^2	Σyz	Σz^2	Σwz
A	107	11087.5888	5299.3084	9084.0187	9144.9346	21972.7290	10697.4206
B	85	7879.5765	5102.1294	7090.9882	8810.2941	18661.6471	8242.7647
C	131	21246.3206	10337.9618	12214.8855	14831.5802	33678.7939	18177.5267
Σ		40213.4859	20739.3996	28389.8924	32786.8089	74313.1700	37117.7120
Total	323	40585.0588	21034.4706	28908.5015	33116.3158	76396.4211	37923.0000

TABLE V

ADJUSTMENT TABLE FOR ANALYSIS OF COVARIANCE— TEACHER-EDUCATION INSTITUTIONS

Group	Denominator	b_1	b_2	$\Sigma z'^2$
A	72,637,194.4788	.615474	.670644	9170.0986
B	29,842,259.5682	.917012	.452314	6854.1838
C	152,647,919.2470	.833277	.450108	13138.1291
Σ	711,533,841.9614	.771113	.525329	29531.8249
Total	730,804,279.7751	.747584	.546949	30897.2463

TABLE VI

WELCH-NAYER TEST FOR HOMOGENEITY OF REGRESSION—
TEACHER-TRAINING COMPARISON

Group	θ''_S	$\log \theta''_S$	$n_S \log \theta''_S$	Additional Data
A	9170.1780	3.9623778		k = 3
B	6854.2344	3.8359590		harmonic mean n_S = 102
C	13138.1291	4.1185309		log L_1 = 9.9980962-10
				L_1 = .996
Σ	29162.5415		1289.5584875	Null Hypothesis:
$\log \Sigma$	4.4648254		1.9537191	P > .05 Accept

TABLE VII

ANALYSIS OF VARIANCE AND COVARIANCE OF FINAL SCORES WITH MENTAL
SCORES AND INITIAL SCORES HELD CONSTANT—
TEACHER-TRAINING COMPARISON

Source of Variation	d.f.	Sum of Squares	Mean Squares	F	Hypothesis
Within Groups	318	29531.8249	92.8674		
Between Groups	2	1365.4214	682.7109	7.351**	Reject:
Total	320	30897.2463			P<.01

TABLE VIII

ADJUSTED MEANS OF GROUPS CLASSIFIED ACCORDING TO TYPE OF COLLEGIATE INSTITUTION WHICH GRANTED THE TEACHER HIS DEGREE

Group	W	$\bar{W}-\bar{W}_T$	$\bar{Y}$	$\bar{Y}-\bar{Y}_T$	$\bar{Z}$	$\bar{Z}''$(adj.)
A	39.72	- .57	38.71	.78	55.51	55.34
B	39.07	-1.22	35.81	-2.12	56.71	58.79
C	41.56	1.27	38.67	.74	61.13	59.74
Total	40.29 ($\bar{W}_T$)		37.93 ($\bar{Y}_T$)		58.11	

W = initial score
Y = mental score
z = final achievement score

TABLE IX

WISHART'S APPLICATION OF THE t TEST TO DIFFERENCES IN ADJUSTED MEANS OF SELECTED GROUPS, COMPARISON 8

Comparison	d.f.	$\bar{Z}_1''-\bar{Z}_2''$	$\frac{R^2A-2LRP+L^2B}{AB-P^2}$	$\sqrt{\frac{V_1(G+N_1+N_2)}{N_1N_2}}$	t	Probability
B - A	190	3.45	.000382	1.412	2.443	.02
C - A	236	4.40	.000139	1.262	3.487	.001
C - B	214	.95	.000295	1.352	.703	.42

V_1 is the adjusted mean square "within"
A is the Σw^2 "within"
P is the Σwy "within"
B is the Σy^2 "within"
Z is the criterion variable
w and y are deviation scores
L equals $\bar{W}_1-\bar{W}_2$ (Table VIII)
R equals $\bar{Y}_1-\bar{Y}_2$ (Table VIII)

in pools. The resulting L_1 value of .995 was greater than the corresponding .05 tabled value for k equal 3 with 103 degrees of freedom. This hypothesis was, therefore, accepted.

The next step was to test the third basic assumption, hypothesis H_3: "The three pooled subsamples were drawn from populations in which the partial regression coefficients of final algebra score upon initial algebra score and mental score within groups were common." This hypothesis was tested by applying the Welch-Nayer test to the adjusted sums of squares within pools. These adjusted sums of squares were obtained by applying the formula:

$$\Sigma z''^2 = \Sigma z^2 - b_1\Sigma yz - b_2\Sigma wz$$

in which $\Sigma z''$ was the adjusted sum of squares of the deviations of final mathematics scores from the mean of the respective pooled groups. The unadjusted sums of within squares was represented by Σz^2. The covariance of final mathematics score with intelligence score and with initial mathematics score were symbolized by Σyz and Σwz, respectively. The regressions of final mathematics score upon intelligence scores and upon initial mathematics scores were represented by b_1 and b_2, respectively. The following formulas were used to obtain b_1 and b_2:

$$b_1 = \frac{\Sigma w^2\Sigma yz - \Sigma wy\Sigma wz}{\Sigma w^2\Sigma y^2 - (\Sigma wy)^2}$$

$$b_2 = \frac{\Sigma y^2\Sigma wz - \Sigma wy\Sigma yz}{\Sigma w^2\Sigma y^2 - (\Sigma wy)^2}$$

Deviation Scores

w = initial mathematics test
y = mental test (Otis)
z = final mathematics test

Table IV contains the variance and covariance sums of the three variables as computed from the means of each pooled group as well as from the grand mean of the combined pools. Table V contains sets of values of the partial coefficients and the adjusted variances. These data were computed from data in Table IV by application of the given formulas. Denominators were included in the table merely to facilitate machine computation.

In the test of H_3 the obtained L_1 was greater than the tabled value at the .05 level for k equal 3 with 102 degrees of freedom (Table VI). H_3 was then accepted and the next step was to test the major hypothesis H_4 by application of the F test. The ratio of the between groups mean square to the within groups mean square was 7.351 (Table VII). This F value exceeded the .01 tabled value for n_1 equal 2 and n_2 equal 318. The major hypothesis was then rejected at the .01 level, and it was concluded that real differences in mean achievement existed among the three categories of classes studied.

The likelihood of real differences among the means of the three groups had been established, but the question of the precise order of thos differences still remained. In order to answer that question the adjusted means for each of the three pooled groups were computed by subtracting from the obtained mean the product of the proper regression coefficient and the algebraic difference between the pooled-group mean and the grand mean of all three groups combined. This correction was made upon the obtained mean to compensate for differences in mean mental score and also for differences in mean initial algebra score among the classes studied. The resulting adjusted-means were 55.34, 58.79, and 59.74 for groups A, B, and C, respectively (Table VIII).

The significance of the differences between adjusted group-means were tested by application of Wishart's[11] modification of the t test. The formula applied was:

$$t = \frac{\bar{Z}_1'' - \bar{Z}_2''}{\sqrt{V_1 (G+M)}}$$

in which $\bar{Z}_1''$ and $\bar{Z}_2''$ are the adjusted means, V_1 is the adjusted sum of "within groups" squares from Table VII, G is the fraction defined in Table IX and M is the sum of the reciprocals of the number of individuals in each group. The results of these tests are recorded in Table IX.

The analysis was then concluded. It had been determined that when the teacher-factors of amount of experience and amount of college mathematical training were controlled by matching, and the pupil-factors of intelligence and initial algebra-achievement were controlled by covariance technique, then, in adjusted-mean mathematical achievement:

1. Classes taught by graduates of private colleges significantly exceeded classes taught by graduates of teachers' colleges. The mean difference of 3.45 points was significant at the .02 level.

2. Classes taught by graduates of state universities significantly exceeded classes taught by graduates of teachers' colleges. The mean difference was 4.40 points, significant at the .001 level.

11. J. Wishart, "Tests of Significance in Analysis of Covariance," Journal of Royal Statistical Society Supplement, III (1932), pp. 79-82.

TABLE X

ASSOCIATION OF SELECTED FACTORS WITH MEAN ALGEBRA ACHIEVEMENT ADJUSTED FOR DIFFERENCES IN INITIAL ACHIEVEMENT AND MENTAL SCORES

***	Groups Compared	Additional Variables Equated	N	F	Group Favored
1.	Boys	a, b, c	229	X	
	Girls		252		
2.	Rural background	a, b, c	182	X	
	Urban background		332		
3.	School enrollment				
	A. Less than 100		101		
	B. From 100-499		116	6.400**	(B)
	C. More than 500		188		
4.	School type				
	A. Four-year	a	212		
	B. Six-year		144	1.072	
5.	Class enrollment				
	A. Less than 20	d	84		
	B. From 20 to 30		147	12.140**	(B)
	C. More than 30		221		
6.	Teacher-mathematical-training	(Groups were heterogeneous as to means)			
7.	Teacher experience				
	A. Less than 2 years	a, e	59		
	B. From 2 to 8 years		157		
	C. More than 8 years		167	7.878**	(C)
8.	Training institution				
	A. Teachers' colleges	a, b, d, e	107		
	B. Private colleges		85		
	C. State universities		131	7.351**	(B, C)
9.	Assignment type				
	A. Identical for all	a, d	230		
	B. Differentiated		225	4.618*	(B)
10.	Daily homework				
	A. Less than 20 minutes	a, d	96	1.584	
	B. More than 40 minutes		105		
11.	Use of applications				
	A. Regularly	a, d	115	3.852*	(A)
	B. Seldom		132		

TABLE X (Continued)

*** Groups Compared	Additional Variables Equated	N	F	Group Favored
12. Use pupil-leadership				
A. Regularly	a, d	71	X	
B. Seldom		123		
13. Review				
A. Frequent	a, d	226	9.126**	(A)
B. Seldom		184		
14. Annual failure-rate				
A. 2% or less	a, d	153	34.999**	(A)
B. 10% or more		146		
15. Use of tests				
A. Frequent	a, d	146	X	
B. Infrequent		143		
16. Supervised study				
A. Less than 20 mins.		74		
B. From 20 to 30 mins.		145	25.526**	(B, C)
C. More than 30 mins.		92		
17. Intra-class grouping	(Groups were heterogeneous as to means)			

a = School size
b = School organization
c = Teacher's amount of experience
d = Teacher's amount of college mathematics
*** Comparison number

X = "within mean square exceeds "between" mean square; not sig.
*Significant at .05 level
**Significant at .01 level

TABLE XI

ASSOCIATION OF SELECTED FACTORS WITH MEAN GEOMETRY ACHIEVEMENT ADJUSTED FOR DIFFERENCES IN INITIAL ACHIEVEMENT AND MENTAL SCORES

*** Groups Compared	Additional Variables Equated	N	F	Group Favored
18. Boys	a, b, c	191	5.786**	(Boys)
Girls		175		
19. College-bound	a, b, c	162		
Terminal		157	X	
20. School enrollment	b			
A. Less than 100		95		
B. From 100 to 500		118	3.326	
C. More than 500		243		
21. School type	a			
A. Three-year		162		
B. Four-year		88	17.696**	(B)
C. Six-year		112		
22. Class-size	a, d, e		1.451	
A. Less than 20		102		
B. From 20 to 30		140		
C. More than 30		104		
23. College mathematics of the teacher	a, d			
A. Less than 2 years		39		
B. More than 2 years		146	X	
24. Teacher experience	a, b, e		2.972	
A. Less than 2 years		28		
B. From 2 to 8 years		170		
C. More than 8 years		69		
25. Training institutions	a, b, d, e		1.059	
A. Teachers' college		137		
B. Private college		114		
C. State university		51		
26. Assignment type	a, b, d, e		3.213	
A. Identical for all		115		
B. Differentiated		135		
27. Daily homework	a, d			
A. Less than 20 min.		60	X	
B. More than 40 min.		90		
28. Use of applications	a, b, d, e			
A. Regularly		54		
B. Seldom		39	X	

TABLE XI (Continued)

*** Groups Compared	Additional Variables Equated	N	F	Group Favored
29. Pupil leadership	a, b, d		1.273	
A. Regularly		108		
B. Seldom		112		
30. Review	a, b, d, e			
A. Frequent		123		
B. Infrequent		170	X	
31. Annual failure-rate	a, b, d, e		2.249	
A. 2% or less		104		
B. 10% or more		83		
32. Use of tests	a, d, e			
A. Frequent		86	4.828*	(A)
B. Infrequent		131		
33. Supervised study	a, b, d, e			
A. Less than 20 min.		172		
B. More than 30 min.		81	4.298*	(B)
34. Intra-class grouping	(Groups were heterogeneous as to means)			
35. Supplementary materials	a, b, d, e			
A. Seldom used		126		
B. Regularly used		116	X	
36. Use of Workbooks	a, b, d, e		2.101	
A. Used a workbook		98		
B. Used no workbook		55		
37. Reasoning problems	(Groups were heterogeneous as to means)			

a = School size
b = School type
d = Teacher's amount of experience
e = Teacher's amount of college mathematics

***Comparison number
X = "within" mean squares exceeded "between" mean squares; not sig.
*Significant at the .05 level

3. Classes taught by graduates of state universities did not significantly exceed classes taught by graduates of private colleges. Test of the observed adjusted mean difference of . 95 points favoring the classes of university-trained teachers led to acceptance of the null hypothesis at the . 42 level. It was probable that this observed difference could be attributed to random sampling errors.

Summary of the Analytical Findings

The results of the thirty-four statistical comparisons employing the technique of analysis of variance and covariance are summarized in Tables X and XI. These comparisons revealed fourteen factors which were significantly associated with mean achievement in elementary algebra, or in plane geometry, or in both.

Pupil Factors. —One of three pupil factors was found to be significantly associated with mathematical achievement: boys exceeded girls in geometry.

School and Class Factors. —The following school and class factors were found to be significantly associated with mathematical achievement:

1. Algebra classes in schools enrolling from 100 to 500 pupils exceeded the achievement of classes in larger and in smaller schools.
2. Geometry classes in four-year schools exceeded the achievement of classes in schools of three-year and six-year organizations.
3. Algebra classes enrolling from 20 to 30 pupils exceeded the achievement of the larger classes as well as the achievement of smaller classes.

Teacher Factors. —The following factors concerning the teacher were found to be significantly associated with achievement in elementary algebra:

1. Classes taught by teachers who had more than eight years of experience exceeded the achievement of classes taught by teachers of less experience. No significant difference was found between the achievement of classes taught by teachers of less than two years experience and the achievement of classes taught by teachers having from two to eight years of experience.
2. Classes of teachers who were graduates of state universities or private colleges exceeded the achievement of classes taught by teachers who were graduates of teachers' colleges.

Teaching Factors. —The following teaching factors were found to be significantly associated with mathematical achievement:

1. Algebra classes in which differentiated assignments were regularly employed exceeded the achievement of classes in which such assignments were seldom employed.
2. Algebra classes in which life applications were regularly studied exceeded the achievement of classes in which life applications were seldom studied.
3. Algebra classes which reviewed the subject matter of the course more than once each month exceeded those classes which reviewed no more than once every six weeks.
4. Algebra classes taught by teachers who usually failed no more than 2 percent of their pupils each year exceeded classes of teachers who usually failed at least 10 percent of their pupils each year.
5. Algebra classes in which supervised study was employed between 20 and 30 minutes per day exceeded the achievement of classes in which less supervised study was used, as well as those classes in which more supervised study was used.
6. Algebra classes in which more than thirty minutes of supervised study was employed exceeded the achievement of classes in which less than twenty minutes were devoted to that activity.
7. Geometry classes which received more than thirty minutes of supervised study exceeded the achievement of classes which received less than twenty minutes of that activity each day.
8. Geometry classes that were tested more than once each week exceeded the achievement of classes which were tested less than once each week.

The results of the statistical analyses revealed that the following factors were not significantly associated with achievement in algebra: sex of the pupil, rural or urban elementary schooling, type of secondary school, amount of assigned homework, extent of pupil-leadership, frequency of tests, and amount of the teacher's mathematical training.

The statistical analyses also indicated a number of investigated factors were not significantly associated with geometry achievement. These factors were: college or terminal plans of the pupil, class-size, teacher-experience, the amount of the teacher's training in college mathematics, the type of collegiate institution from which the teacher had been graduated, type of assignment, amount of assigned homework, use of applications, use of pupil-leadership, frequency of reviews, annual failure rate, use of supplementary materials, and the use of a workbook.

The results of the various comparisons were consistent in algebra and geometry with respect to revealed significant association of school enrollment size, supervised study practices, and with respect to the lack of significant association noted between mathematical achievement and the three factors of: amount of daily homework assigned, use of pupil-leadership, and the amount of college mathematics studied by the teacher.

Summary of the Descriptive Findings

The ninety-item schedule employed to collect

TABLE XII

TEACHING-PRACTICE PERCENTAGES REVEALED BY THE TEACHER-SCHEDULE

Teaching Practice	67 teachers of Elementary Algebra				68 teachers of Plane Geometry			
	Seldom	Freq.	Reg.	NR*	Seldom	Freq.	Reg.	NR*
Assignments								
Turned in daily	1	25	74	0	9	41	50	0
Non-textual materials	52	33	3	12	37	47	1	15
Studied in class	9	33	57	1	7	37	56	0
Undifferentiated	25	31	43	1	21	26	52	1
Subject matter presentation								
Lecture and recitation	18	48	33	1	29	43	21	7
Pupil-discovery	15	48	9	28	15	63	9	13
Text-recitation	43	28	8	21	40	31	12	18
Student demonstration	31	38	9	21	24	51	13	12
Classroom activity								
Drill	10	39	49	1	19	47	24	10
Memorization	49	25	12	13	26	41	21	12
Practical applications	13	55	16	15	15	57	15	13
Appreciations (art, history)	46	25	4	24	46	32	6	16
Reasoning applications	15	53	13	18	15	57	15	13
Functional thinking	22	49	10	18	22	57	10	10
Accommodations for differences								
Homogeneous class sections	63	4	16	16	66	7	7	19
Grouping within classes	54	27	3	15	53	16	9	22
Ability-level assignments	27	40	19	13	27	47	10	16
Marking adjustments	31	16	38	13	40	19	31	10

*No response

TABLE XIII

QUARTILE POINTS IN DISTRIBUTIONS OF SELECTED TEACHING PRACTICES REVEALED BY THE TEACHER-SCHEDULE

Teaching Practice	Q_1	Q_2	Q_3
Number of years of teaching experience	3	8	10
Algebra class enrollment	21	26	32
Geometry class enrollment	13	20	27
Number of different subjects taught	2	4	5
Number of mathematics classes taught daily	2	3	4
Number of daily classes outside field of math	0	1	2
Approximate minutes of daily homework assigned	20	30	39
Approximate minutes of daily supervised study	21	25	32
Average percentage of annual failures	1	2	5
Average percentage of annual A's and B's	9	18	28
Average hours of marking and recording per week	3	5	10
Average weekly hours of instructional planning	3	5	5
Average weekly hours of miscellaneous duties	2	3	5
Scheduled classes and supervision per week (hours)	23	28	30
Total load in hours per week	38	42	47

data concerning the status of mathematics instruction in Minnesota schools was returned by 123 of the 130 teachers participating in the study at the end of the seventh school month. Chi-square tests indicated that the teachers who responded did not differ significantly from the non-respondents with respect to their professional training and experience. The fact that the respondents were teaching in randomly selected representative schools further enhances the likelihood that the findings based on this sample were valid for the population.

The data collected were employed primarily as a basis for classification of the groups compared in the foregoing statistical analyses. The secondary function of these data was that of determining the status concerning working conditions, instructional practices, and professionalism of Minnesota teachers of mathematics. The findings of the latter phase of the study are summarized in Tables XII and XIII.

Examination of these data revealed that wide variability existed with respect to many of the factors studied. The average practices compared favorably with standards recommended by educational authorities. However, comparison of the lower extreme of the distribution with minimum standards revealed that some teachers deficient in professional qualifications were using outmoded methods and inadequate materials in the instruction of an overload of excessively large classes. The following brief summary places emphasis upon those factors which teachers and supervisors might wish to improve:

1. Professional qualifications. —The bachelor's degree was held by 97 percent of the respondent teachers, while the master's degree was held by 21 percent. However, 27 percent of the respondents had studied mathematics for less than two years in college, and 2 percent were teaching the subject with an academic background of less than six semester hours of college mathematics.

2. Teaching load. —The average daily teaching load was five classes of approximately twenty-five pupils each. The average number of different subjects taught each day was four, including one outside the field of mathematics. Twenty-nine percent of the respondents had fewer than three preparations per day, while 27 percent taught five different subjects daily, and 18 percent taught in three or more fields in addition to mathematics. Thirty-one percent of the classes enrolled more than thirty pupils.

3. Instructional planning. —The average respondent spent five hours per week scoring and recording papers and an equal number of hours on instructional planning. On the other hand, some teachers spent more time on the scoring and recording of papers than on instructional planning. Nine percent devoted, at most, one hour per week to planning; whereas, 27 percent spent two hours per day on paper work.

4. Professionalism. —Ninety percent of the respondents had attended at least one state or regional educational convention in the three-year period 1945-1948. Thirty-five percent reported attending at least one session of summer school, a workshop, or other in-service training activity. However, in the same three-year period, less than 20 percent had attended a state or regional mathematics conference; less than 15 percent had subscribed to either the Mathematics Teacher or School Science and Mathematics; and only 6 percent belonged to a local mathematics organization.

5. Availability of supplementary materials. —Fifty-eight percent of the respondents reported the availability of fewer than ten items of supplementary printed materials, and 50 percent reported that such materials were used only once a month, or less. Only 30 percent had used any films or film-strips in mathematics.

6. Recognition of individual differences. —Approximately 60 percent of the responding teachers assigned identical problems to all pupils each day, and 40 percent seldom assigned materials outside of the textbook. Remedial work as follow-up to tests was used regularly by less than 40 percent of the respondents.

7. Application of psychological principles. —Only 10 percent of the respondents regularly employed pupil-discovery, pupil demonstrations, or pupil-led discussions. More than 40 percent regularly employed drill, and 18 percent regularly emphasized memorization. Yet, less than 20 percent regularly planned work emphasizing life applications, appreciations, reasoning in everyday life, or functional thinking. Forty percent of the respondents employed review no more than once every six weeks. Ten percent of the respondents gave tests no more than once each month, and the same proportion annually failed 10 percent of their pupils.

Implications of the Study

Implications of the Descriptive Findings. —From the results of the descriptive phase of the study, it may be implied that:

1. If all secondary school pupils in Minnesota are to have opportunity for mathematical instruction equal to that recommended by educational authorities, it will be necessary to improve the present conditions with respect to (a) the academic training and professionalization of some teachers, (b) the instructional techniques employed, and (c) the educational and psychological principles observed.

2. If all Minnesota teachers of secondary school mathematics are to have the opportunity to teach

under conditions equal to those recommended by educational authorities, it will be necessary to improve the present status with respect to the (a) number and kind of preparations required per day, (b) size of the pupil-day load, and (c) availability of materials of instruction.

Implications of the Analytical Findings. —In the statistical and analytical phase of the study, the data revealed factors that were significantly associated with achievement in mathematics. Utilization of this information, in whole or in part, by teachers, supervisors, administrators, and board members could aid in the improvement of mathematics instruction. Specifically, it may be said that, with particular application in the average algebra class, pupil-learning would be enhanced:

1. if assignments would be differentiated.
2. if regular study of life applications would be employed
3. if reviews would be made more frequently
4. if more supervised study were used (optimum length was between 20 and 30 minutes in algebra and more than 30 minutes in geometry)
5. if tests were to be used more frequently (optimum use was more than once per week in geometry, but undetermined in algebra)
6. if annual failure-rate would be made less severe.

The various boards of education, as well as superintendents and principals might make valuable use of the finding that:

1. Teachers of more than eight years' experience were more successful in teaching algebra than less-experienced teachers.

2. The optimum size class was between 20 and 30 pupils.

3. Schools may be too large or too small to provide optimum educational facilities. Less mathematical learning took place in schools under an enrollment of 100 and over an enrollment of 500 than took place in the intermediate-sized schools (100-500 pupils).

4. The four-year organization was better than the six-year organization with respect to geometry achievement.

5. Graduates of state universities and private colleges were more successful in algebra teaching than were graduates of teachers' colleges.

Interpretation and Use of Analytical Survey Results

In this study, interpretation of causation frequently attributed to factors characterized by more association was stringently avoided. Distinction was made between the type of deduction that can be made from survey data and the type appropriate from experimental data. It was recognized that in order to determine causally the magnitude of the effect of any factor, controlled experiments are needed. Deductions of the magnitude of effects cannot be made from survey data. However, clearly definable extraneous factors can be identified and their effect controlled in an analytic survey. Furthermore, the relationship of certain educational factors to achievement can be determined as was shown in the analysis. The investigator recognized that in any survey there will always be other factors, unascertained or perhaps unascertainable, which have not been taken into account. Nevertheless, analytical surveys are of value when widespread experiments are difficult or impossible, or as regarded here, for over-all evaluations of a state's mathematics program. The methodology, thus employed, will also serve to identify factors that are promising for investigation by experiment to follow.

Recommendations

It is recommended that results of educational research be made more directly available to those who establish educational policies and to those who supervise and undertake their execution. In that manner, improvement in instruction will be facilitated to the benefit of the pupil. It is further recommended that subsidies be granted by governmental and private institutions so that experienced teachers, professional investigators, and competent graduate students may be encouraged to undertake the further research of a causal-comparative and experimental nature which is required to establish mathematical instruction upon a scientific basis and assure its continuous improvement.

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THE SYMMETRICAL IDEMPOTENT MATRIX IN FACTOR ANALYSIS*

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It is of interest to note the occurrence in factor analysis of the symmetrical idempotent matrix and to examine the role it may play as a multiplier of other matrices. For example, Holzinger and Harman's method of relating two different factorial solutions of the same correlation matrix[1] may be formulated, alternatively, in terms of the symmetrical idempotent matrix as a multiplication unit for certain factor matrices. Tucker's semi-analytic method of rotating to a simple structure,[2] which utilizes the same equation as that given by Holzinger and Harman, also may be formulated in these terms, providing the effect of using the symmetrical idempotent as a multiplier of matrices for which it is not a unit for multiplication is noted. The Holzinger-Harman treatment is concerned with the problem of solving for a transformation matrix that rotates one given factorial solution into a second, and given, one. Tucker's problem differs, in that the second solution is not given but is constructed from an initial arbitrary matrix through a series of approximations; he therefore introduces the requirement of a best fit in the least squares sense and shows that the type of matric equation employed meets this requirement. Somewhat earlier, Horst had proposed a rotation method, using similar principles, that involved the actual calculation of a symmetrical idempotent matrix and the use of its entries as a guide to the rotation process.[3] These examples suggest that a review of the idempotent as a particular type of matric element may aid in clarifying certain algebraic principles common to these several papers and in outlining certain applications of these principles in factor analysis.

As a preliminary step, consider the equation:

$$FT = C, \tag{1}$$

in which F and C are given and T is to be solved for. The solution is obvious if F is non-singular. If F is singular, however, Holzinger and Harman suggest the solution:

$$T = (F'F)^{-1}F'C, \tag{2}$$

which is accomplished by pre-multiplying both sides of (1) by $(F'F)^{-1}F'$.[4] A problem they do not treat explicitly is how to reconcile equation (1) with equation (3), below, which is a pre-multiplication of both sides of (2) by F.

$$FT = \left[F(F'F)^{-1}F'\right] C. \tag{3}$$

As will be shown later, the product within brackets in (3) is a symmetrical idempotent matrix, and a necessary and sufficient condition for (1) and (3) to be identical is that the idempotent be a multiplication unit for matrix C.

Now rewrite equation (1) as an approximation, i.e., that the product FT yield not C, but an approximation to C.

$$FT \backsim C \tag{1a}$$

Let this approximation be a best fit in the least

* Appreciation is expressed to Professor Richard Bruck of the Department of Mathematics, University of Wisconsin, for reading portions of this paper.

1. Karl J. Holzinger and Harry H. Harman, Factor Analysis. (Chicago: University of Chicago Press, 1941), pp. 289-292.

2. Ledyard Tucker, "A Semi-Analytical Method of Factorial Rotation to Simple Structure," Psychometrika, IX (March 1944), pp. 43-68.

3. Paul Horst, "A Non-Graphical Method for Transforming an Arbitrary Factor Matrix into a Simple Structure Factor Matrix," Psychometrika, VI (April 1941), pp. 79-99.

4. Holzinger and Harman, op. cit., p. 290. They specify F as a rectangular matrix of order n by m. They necessarily assume that (F'F) is non-singular. Also see Louis Guttman, "General Theory and Methods for Matric Factoring," Psychometrika, IX (March 1944), pp. 11-12, for a critical comment on the Holzinger-Harman equation.

squares sense. Tucker shows[5] that the necessary requirement is that:

$$F'C = F'FT. \qquad (4)$$

The value of T that satisfies (4) is that given in equation (2), above, since

$$F'C = F'F(F'F)^{-1}F'C = F'C. \qquad (5)$$

The conclusion is that the expression on the right of (3) is the least squares approximation to C when (1a) holds. Apparently, then, the symmetrical idempotent matrix constructed from the given matrix F operates either as a pre-multiplication unit for C, in which case (1) is satisfied, or as a pre-multiplier of C that yields a least squares approximation of C, as in (1a).

These preliminaries in effect set two topics for discussion. One is, what conditions on the matrix F are required for the construction of the symmetrical idempotent? The second, and more extensive, is, what characteristics of the matrix C determine whether (1) or (1a) holds? The attempt to answer these questions will demand review of certain well-known algebraic concepts, such as that of the idempotent, the group, and the ring. In this review the conventional notation of factor analysis will be employed, and at a number of points attention will be centered on the symmetrical matrices that commonly constitute the data taken for analysis. The restriction that the elements of the matrices under discussion belong to a given field will be imposed. As practice indicates, this is taken to be the field of real numbers by the factor analyst.

Construction of the Symmetrical Idempotent

Postulate a square matrix of order n, such that:

$$A = A^2. \qquad (6)$$

Two solutions are immediately apparent; one is the null matrix, O, and the other the identity matrix of order n, I. Let λ designate the characteristic roots of a matrix; it is then evident that (6) requires that $\lambda^2 - \lambda = 0$. The function $\lambda = 0$ is the minimum function of the null matrix, and is an elementary divisor of $\lambda^2 - \lambda = 0$. Similarly, the function $\lambda - 1 = 0$, which is the minimum function of the identity matrix, is an elementary divisor of $\lambda^2 - \lambda = 0$. Both these matrices therefore satisfy the requirement of (6). A third type of matrix, differing from the null and the identity matrix, has as its minimum function $\lambda^2 - \lambda = 0$ and also satisfied (6). It is called the idempotent matrix.[6]

Given this definition, the problem is one of noting what is required to construct a matrix that will satisfy (6). Assume a rectangular matrix, F, of order n by m and rank m ($n \geq m > 0$). This implies that F may not be a zero matrix, and that it must consist of exactly m linearly independent columns. Utilizing F, construct the matrix:

$$F(F'F)^{-1}F' = A \qquad (7)$$

and note that this matrix satisfies (6), since

$$\left[F(F'F)^{-1}F'\right]\left[F(F'F)^{-1}F'\right] = F(F'F)^{-1}F'. \qquad (8)$$

The A of (7) has certain evident characteristics. It is square, of order n, and symmetrical. It cannot be the identity matrix of order n, since F is singular. Also, it cannot be the null matrix; this will follow from a demonstration of the rank of A.

Since F was required to have m linearly independent columns, it may be represented as m independent column vectors, Φ_m. The multiplication (F'F) may be written:

$$\begin{Vmatrix} O'_1 \\ O'_2 \\ \cdot \\ \cdot \\ O'_m \end{Vmatrix} \cdot \begin{Vmatrix} O_1 & O_2 & \cdots & O_m \end{Vmatrix} = \begin{Vmatrix} O'_1O_1 & O'_1O_2 & \cdots & O'_1O_m \\ O'_2O_1 & O'_2O_2 & \cdots & O'_2O_m \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ O'_mO_1 & O'_mO_2 & \cdots & O'_mO_m \end{Vmatrix}$$

5. Tucker, op. cit., pp. 50-51. As he points out, a similar proof had been given earlier.

6. A. Adrian Albert, Modern Higher Algebra (Chicago: University of Chicago Press, 1937), p. 88.

From this form it is clear that $(F'F)$ is non-singular of order m, since linear dependence of the arrays of the product would imply linear dependence of the vectors, Φ_m, which is contrary to the initial restriction. Since $(F'F)$ is non-singular its inverse exists and the construction of (7) is shown to be possible.

Since $(F'F)$ is non-singular, pre-multiplying F' by the inverse, $(F'F)^{-1}$, will not alter its rank.[7] Therefore, the product $(F'F)^{-1}F'$ may be represented as m independent row vectors, say, Ψ_m. The rectangular matrix F may be made into a square matrix of order n by adding n-m columns of zeroes. This matrix may then be put into its Hermite canonical form[8] and the rows permuted by pre-multiplying it by a non-singular matrix of order n, say, Q. This form is:

$$QF = \left\| \begin{array}{ccccc|c} 1 & 0 & 0 & . & 0 & \\ a_{21} & 1 & 0 & . & 0 & \\ a_{31} & a_{32} & 1 & . & 0 & 0_{m,\,n-m} \\ . & . & . & & . & \\ a_{m_1} & a_{m_2} & a_{m_3} & . & 1 & \\ \hline & 0_{n-m,\,m} & & & & 0_{n-m,\,n-m} \end{array} \right\|$$

Post-multiplying QF by the product $(F'F)^{-1}F'$, i.e., by a matrix of m row vectors, gives an n by n matrix whose non-zero row vectors are exactly m in number and are of the form:

$$\Psi_1,$$

$$a_{21}\Psi_1 + \Psi_2,$$

$$a_{31}\Psi_1 + a_{32}\Psi_2 + \Psi_3$$

$$a_{m_1}\Psi_1 + a_{m_2}\Psi_2 + a_{m_3}\Psi_3 + \ldots + \Psi_m.$$

This form makes clear the linear independence of these m vectors, since their linear dependence would require linear dependence of the vectors Ψ_m, which is contrary to the restriction above. It may be concluded, then, that the row rank and consequently the rank of the product $Q[F(F'F)^{-1}F']$ is exactly m. But this product represents the pre-multiplication of the matrix A of (7) by a non-singular matrix of order n. Since pre-multiplying a matrix by a non-singular matrix does not alter its rank, it follows that A is of rank m. In general, then, any n by n matrix of the form FXF' is of rank m providing F has n rows and m linearly independent columns and X is any m by m non-singular matrix.

Since the matrix A of (7) is square, is of order n and rank m, and satisfies (6), it may be specified as an idempotent matrix with minimum function $\lambda^2 - \lambda = 0$. We also note that A is symmetrical in form.[9] What has been done, then, is to outline a method of constructing a symmetrical idempotent, given a rectangular matrix of order n by m and rank m. The number of rows of the rectangular matrix determines the order of the idempotent, and the rank of the rectangular matrix determines the rank of the idempotent. This method may be used to construct a symmetrical idempotent of any desired order, n, and rank, m, within the restriction that $n>m>0$. As special

7. Cyrus C. MacDuffee, Vectors and Matrices, Carus Mathematical Monographs, No. 7 (The Mathematical Association of America, 1943), p. 34.
8. MacDuffee, op. cit., pp. 35-38.
9. Non-symmetrical idempotents also exist. Wedderburn gives as an example:

$$\left\| \begin{array}{cc} 4 & -2 \\ 6 & -3 \end{array} \right\|$$

J. H. M. Wedderburn, Lectures on Matrices, Colloquium Publications, Vol. XVIII (New York: American Mathematical Society, 1934), p. 7.

cases of (7), we note that A = 0 if F is a zero matrix, and A = I of order n if F is n by n and non-singular.

A Multiplicative Group of Singular Matrices

The symmetrical idempotent matrix may be shown to be the unique right and left unit for multiplication within a multiplicative group of singular matrices. As a first step, note that given F defined as above, a symmetrical matrix of order n and rank m may be constructed, thus:

$$FF' = R \tag{9}$$

The symmetrical idempotent, A, is both a right and left unit for multiplication of R, i. e.,

$$AR = RA = R \tag{10}$$

as can be verified by direct multiplication using (7) and (9). F may be called a factor of R.

It is well known that post-multiplying F by any m by m orthogonal matrix yields a "new" factor matrix, FT_O, such that,

$$FT_OT_O'F' = FI_mF' = R \tag{11}$$

It also is known that given F, pairs of so-called oblique factors that also reproduce R may be constructed by employing non-orthogonal, non-singular m by m matrices as transformation matrices. In general, let

$$\left.\begin{aligned} FT_j &= S_j \\ \text{and} \quad S_j(T_j'T_j)^{-1} &= P_j \end{aligned}\right\} \tag{12}$$

Then:

$$P_jS_j' = S_jP_j' = R. \tag{13}$$

By allowing T_j to range over all possible m by m non-singular matrices, there results an indefinite number of factor matrices FT_j that reproduce R by the operation of (11) or (13), corresponding to T_j orthogonal or not. An important question is the effect of transforming F in this manner upon the symmetrical idempotent matrix.

Substitution of any FT_j for F in (7) gives:

$$FT(T_j'F'FT_j)^{-1}T_j'F' = F(F'F)^{-1}F', \tag{14}$$

providing only that T_j is non-singular as was specified. Therefore the symmetrical idempotent of (7) is invariant with respect to these transformations[10] of F. By allowing T_j to range as indicated, the expression FT_j designates all n by m matrices that may be used in the manner of (7) to construct the specified idempotent A. However, FT_j does not designate all n by m matrices of rank m. For example, pre-multiply F by a non-singular matrix Q of order n to yield such a matrix. Then:

$$QF(F'Q'QF)^{-1}F'Q' \neq F(F'F)^{-1}F'. \tag{15}$$

The matrix on the left of (15) also is a symmetrical idempotent of order n and rank m, but as (15) shows it is not identical with A and is not a multiplication unit for the R of (9). It will be recalled that equation (3) posed the problem of the characteristics of any matrix for which the symmetrical idempotent is a unit for multiplication. One might suspect that for the matrix to be of the same order and rank as the idempotent would be sufficient. This demonstration, however, indicates that this is not true.

Given the matrix F, it is possible to construct the idempotent A and the symmetric matrix R for which A is a multiplication unit. The problem is to construct additional n by n matrices for which A is a multiplication unit and to show they constitute the elements of a multiplicative group. By employing the m by m non-singular matrices T_j as transformations of F, this construction may be accomplished. For example,

$$FT_jT_j'F' = R_j, \tag{16}$$

where R_j is symmetrical but distinct from R providing T_j is not orthogonal. Also,

$$\left.\begin{aligned} FT_jT_k'F' &= W \\ \text{and} \quad FT_kT_j'F' &= W' \end{aligned}\right\} \tag{17}$$

where W is non-symmetrical and necessarily distinct from R or R_j. Any R_j, including R which is the special case of T_j orthogonal in (16), satisfies (10). Similarly, direct multiplication verifies that for any W,

$$\left.\begin{aligned} AW &= WA = W \\ AW' &= W'A = W' \end{aligned}\right\} \tag{18}$$

In general, then, any n by n matrix of the form FXF', with X m by m and non-singular, is transformed into itself by either pre- or post-multiplication by the symmetrical idempotent constructed from F. We will specify all such matrices as sub-set of the square matrices of order n and rank m; it obviously is a sub-set, since with Q any non-singular matrix of order n, the similar

10. These transformations constitute a multiplicative group of non-singular matrices.

matrix $QFXF'Q^{-1}$ is also of order n and rank m but is not transformed into itself by A. Further, the matrix A is a member of this sub-set, since $(F'F)^{-1}$ is non-singular of order m and thus the idempotent A is itself of the form FXF'.

For a set of square matrices to be the elements of a multiplicative group, three postulates must be satisfied.[11] One is that the associative law hold for multiplication of the elements, i.e.,

$$B(CD) = (BC)D. \tag{19}$$

This is generally true for matrices. A second postulate is that the set be closed with respect to multiplication, i.e., that

$$BC = D \tag{20}$$

with D an element of the set for every B and C. Since multiplication of matrices ordinarily is not commutative, BC is generally distinct from CB, and this requirement is that both BC and CB yield an element of the set for every B and C. Taking as the set all square matrices of order n and rank m of the form FXF', in which X is order m and non-singular, the product of any two such matrices gives:

$$(FX_jF')(FX_kF') = F\left[X_j(F'F)X_k\right]F' \tag{21}$$

It has been shown that (F'F) is of order m and non-singular; therefore the product within brackets in (21) reduces to some X of order m and non-singular, and the matrix on the right of (21) consequently is a member of the set. Closure with respect to multiplication is thus established.

A third requirement is that there be elements of the set, Y and Z, such that:

$$BY = C \tag{22}$$

and

$$ZB = C \tag{23}$$

for every B and C within the set. Since attention has been centered on the idempotent A, let us first show it the unique unit element for multiplication within the set[12], and then show that these equations are satisfied. Any unit for multiplication within the multiplicative group must satisfy

$$ABA = B \tag{24}$$

for every B. Analyzing the matrices A and B into the general form FXF', this requirement is that

$$(FX_uF')(FX_jF')(FX_uF') = FX_jF', \tag{25}$$

or

$$F\left[X_u(F'F)X_j(F'F)X_u\right]F' = FX_jF'. \tag{26}$$

Since the m by m matrices within brackets in (26) are all non-singular, one solution for X_u is $(F'F)^{-1}$, which gives as a unit element $A = F(F'F)^{-1}F'$. To show A unique, assume another unit, U. Then since A is a unit, AU = U. But if U is also a unit, AU = A. Therefore, A = U, and A is unique. Therefore, if in equations (22) and (23) B = C, the solution for both Y and Z is the symmetrical idempotent A.

If $B \neq C$, the solutions for Y and Z are as follows. Rewrite (22) as

$$(FX_jF')(FX_mF') = FX_kF'. \tag{27}$$

Then:

$$Y = F\left[(F'F)^{-1}X_j^{-1}X_k\right]F' \tag{28}$$

and

$$FX_jF'F\left[(F'F)^{-1}X_j^{-1}X_k\right]F' = FX_kF' \tag{29}$$

The matrix within brackets in (28) is the product of three non-singular matrices of order m, and consequently is itself non-singular of order m. Y, therefore, is a member of the group. Similarly,

$$Z = F\left[X_kX_j^{-1}(F'F)^{-1}\right]F', \tag{30}$$

and is a member of the group. An arbitrary notation may be employed to designate these solutions:

$$Y = B^{-I}C \tag{31}$$

and

$$Z = CB^{-I}, \tag{32}$$

where

$$B^{-I} = F\left[(F'F)^{-1}X_j^{-1}(F'F)^{-1}\right]F' \tag{33}$$

and thus

$$BB^{-I} = B^{-I}B = F(F'F)^{-1}F' = A. \tag{34}$$

We have, then, a multiplicative group of singular matrices with a unique multiplication unit, A, and the "inverse", B^{-I}, of every matrix in the group.

A Ring of Singular Matrices

An important characteristic of the symmetrical idempotent is that in addition to being the unique

11. Albert, op. cit., pp. 7-9. Note that it was previously stated that the set of all non-singular matrices of a given order forms a multiplicative group. It is a simple task to show the postulates are satisfied for such a case.

12. The matrix I of order n is excluded from the set, since it is of rank n.

multiplication unit within a multiplicative group of singular matrices, it also is a unit for multiplication of the elements of an additive group of singular matrices, some of which are the elements of the multiplicative group. The set of all square matrices of order n with elements in a given field form a total matric ring, say, M_n. The null matrix of order n, which is a unit for addition, and the identity matrix of order n, which is a unit for multiplication, are members of this total matric ring. We may pre- and post-multiply each element of this ring by A, thus forming a set of matrices, AM_nA.[13] The result is to give a set of square matrices, all of rank m or less, that form an additive group. Since the ring, M_n, is closed with respect to addition, it follows that

$$AM_jA + AM_kA = A(M_j + M_k)A \qquad (35)$$

with every $A(M_j + M_k)A$ a member of this additive group. Since M_n contains the unit for addition and the inverse with respect to addition of every element, it follows that

$$AM_jA + A(-M_j)A = AOA = O \qquad (36)$$

for every M_j, with AM_jA, $A(-M_j)A$, and the unit element members of this additive group. Consequently, for every AM_jA and AM_kA there exists a solution AM_xA, for the equation,

$$AM_jA + AM_xA = AM_kA, \qquad (37)$$

namely, $$AM_xA = A(M_k - M_j)A, \qquad (38)$$

which is a member of the group. The operation of addition is associative; it also is commutative, and so the additive group is commutative or abelian.

Since the elements of the total matric ring are closed with respect to multiplication, and since the matrix A is an element in M_n, the matrices of the form AM_nA also are closed with respect to multiplication. That is,

$$(AM_jA)(AM_kA) = A\left[\, M_jAM_k \,\right]A, \qquad (39)$$

with the product within brackets some element of the total matric ring, and consequently the term on the right some element of AM_nA. We may therefore define AM_nA as a sub-ring of singular matrices, since: the elements constitute a commutative group with respect to addition; the elements are closed with respect to multiplication; multiplication is associative and is distributive with respect to addition.[14] The unit element for multiplication in this sub-ring is the symmetrical idempotent A, i.e., AIA. Further, the elements of AM_nA are all of the form FXF', where X is square, of order m, but may have any rank not exceeding m. Evidently, then, the multiplicative group discussed in the preceding section constitutes all the rank m elements of this sub-ring.

Analysis of the Initial Equations

Return now to equations (1) and (3). Note that whenever the equations are identical, i.e., AC = C, it follows that CC' is some (symmetrical) member, AM_jA, of this sub-ring. It can also be shown[15] that if, and only if, CC' is a member of the sub-ring will the equations be identical, providing the elements of the matrices are real numbers. This is the general case, including the case of CC' a member of the multiplicative group. That C may be of rank less than m, the rank of F, yet the two equations be identical, is the interesting but perhaps unexpected result. The character of least squares approximations to C has now also been described. They are of the form AC, and a consequence is that every ACC'A is a member of this sub-ring, since C necessarily has n rows and therefore any CC' is an element of the total matric ring, M_n. Briefly, then, if AC = C, CC' is a member of the sub-ring AM_nA and must be of rank $\geq$ m; also, with C consisting of real numbers, whenever CC' is a member of AM_nA, AC must equal C. If AC is not equal to C, then CC' must be in M_n but not in the sub-ring AM_nA; and, considering only real numbers, for any CC' in M_n but not in AM_nA, AC is not equal to C. A least squares approximation to C is such that ACC'A is always a member of AM_nA. If we view C as a factor of some M_j, in M_n but not in AM_nA, then the least squares approximation of F to C may be viewed as the analogous factor of

13. See MacDuffee, op. cit., pp. 67-68. The concept of the decomposition of an algebra is also relevant; see Albert, op. cit., pp. 88-89.

14. MacDuffee, op. cit., p. 178. It can readily be shown that this latter requirement is met.

15. Put the idempotent into its canonical form and let C be a super matrix consisting of four matrices: Z_{11}, Z_{12}, Z_{21}, Z_{22}. With the elements of C real, CC' a member of the sub-ring requires that Z_{21} and Z_{22} be zero matrices. Then AC = C. Further, with CC' not a member of the sub-ring, AC $\neq$ C.

AM_jA, which is in AM_nA.

Conclusion

The primary intention of this paper has been to call attention to the occurrence in factor analysis of the symmetrical idempotent matrix and, using it as center of attention, to review the concepts of the multiplicative group of singular matrices and the sub-ring AM_nA. Consideration of these concepts in relation to some of the problems of factor analysis suggests a number of applications. A few of these will be outlined briefly without detailed proof or discussion in this concluding section.

The concept of the sub-ring AM_nA gives a formulation of least squares factors. Assume a matrix R, of unknown rank, that is precisely factorable by some orthogonal factor matrix X, i. e., $XX' = R$. Choose a factoring method and develop the orthogonal factor matrix F, consisting of as many columns as the criteria that are being employed indicate. A residual matrix is then given by $R - FF'$. If this is not null, it may be desired to transform F into a least squares approximation to the unknown X. Set

$$FY' = ARA, \tag{40}$$

where A is an idempotent constructed from F by (7). Then

$$Y' = (F'F)^{-1}F'RA. \tag{41}$$

Set $$FZ = Y. \tag{42}$$

Then $$Z = (F'F)^{-1}F'RF(F'F)^{-1}. \tag{43}$$

It will be observed that the definition of Z is the same if FY' is set as an approximation to R in (40) and the least squares approximation employed.

We note that what is being done here is to consider R a member of the total matric ring M_n but not of the sub-ring AM_nA, and then to identify within the sub-ring the matrix ARA, which is a least squares approximation to R. The character of F imposes the basic restriction, since it determines the symmetric idempotent A and thus determines which matrices are to be segregated into the sub-ring. Clearly, when judged by other criteria, ARA may be a poor approximation to R; it is, however, a best approximation in the least squares sense within the limitations imposed by the procedures used to develop F. Also, two different residual matrices are being defined. One is $R - FF'$, which is the conventional one; the other is $R - ARA$. It will always be true that

$$F(R - ARA)F' = O, \tag{44}$$

but with F singular this is of course no guarantee that $R - ARA$ is a null matrix.

The matrix Z of (43) may be an identity. If so, $F = Y$, and the two types of residual matrix are identical. Consider only the first column F_1 of a factor matrix developed by some method. If there exists a real factor matrix X, such that $XX' = R$, that equals or is some orthogonal rotation of some F having F_1 as its first column, it follows that

$$F_1'RF_1 \geq (F_1'F_1)^2. \tag{45}$$

Occurrence of the equality implies that the factoring must be continued in such a way that at any stage $F'F$ equals a diagonal matrix, and that when so continued Z will be an identity at any stage, including the first, and thus the method may be characterized as a least squares one. The principal-axis method of factoring will readily be identified as such a method.

Occurrence of the inequality in (45) implies the use of a method such that $F'F$ is not a diagonal. Since the single value given by $F_1'RF_1$ does not change with the extraction of successive factors, this relationship may suggest a rough guide to the number of factors to extract, i. e., continue to extract factors until the first row and first column entry of $(F'F)^2$ exceeds the value given by $F_1'RF_1$. It has been verified that with hypothetical problems that are precisely factorable by a method such as the centroid or the bi-factor, $(F'F)^2$ exactly equals $F'RF$ after m factors. It also has been noted that $F_1'RF_1$ may with actual problems be less than $(F_1'F_1)^2$. This would appear to constitute a rejection of the hypothesis that R is precisely factorable by a real X that is an orthogonal rotation of some F having this F_1 as its first column.

Any singular matrix expressible in the form FF' is a member of some multiplicative group of singular matrices. The problem of factoring such a matrix may be viewed as the problem of identifying any rectangular matrix that will, by the operation of (7), give the symmetrical idempotent A that is the unit for multiplication within this group. Any such rectangular matrix may then be put into the more familiar form either of orthogonal or oblique factor matrices for which the factors are taken at unit length. Conversely, a criterion for the "perfect" factoring of any such matrix is given by:

$$R = RA \tag{46}$$

or $$R(I - A) = O \tag{47}$$

where $(I - A)$ also is idempotent, but of rank $n - m$ and such that it annihilates the group of A. Since $(I - A)$ is idempotent, its centroid factors

are also its principal-axis factors; and since the non-zero characteristic roots of any idempotent are all unity, the centroid factors of (I - A), say, G_1, are such that

$$G_1'G_1 = I_{n-m}. \tag{48}$$

Therefore, we have from (47)

$$RG_1 = O, \tag{49}$$

in which G_1 consists of (n - m) columns. More than one fundamental set of solutions[16] exists for (49). However, as (48) indicates, the n - m columns of G_1 must be columns of some orthogonal matrix of order n. By imposing this restriction it is possible to select a fundamental set that satisfies both (48) and (49).

The form of the solution of the homogeneous equations of (49) may be of interest. It is necessary that m, the rank of R, be known and that R be pre-multiplied by some elementary matrix to yield m linearly independent rows. Let QR designate this m by n matrix with the rank of R. Partition QR into an m by m non-singular matrix, V, and an m by (n - m) singular matrix, W. Let Y be any non-singular matrix of order (n - m). Then any fundamental set of solutions of (49) is given by:

$$\left\| \begin{matrix} V^{-1}(-W)Y \\ Y \end{matrix} \right\| \tag{50}$$

The restriction of (48) demands that

$$I = Y' \left[(-W')(VV')^{-1}(-W)+I \right] Y \tag{51}$$

where each I is of order n - m, or,

$$YY' = \left[(-W')(VV')^{-1}(-W)+I \right]^{-1} \tag{52}$$

Take Y as an orthogonal factor of the matrix on the right of (52) to construct the required G_1 by way of (50). Then direct multiplication shows not only that (48) is satisfied by this G_1 but also that G_1G_1' is idempotent and annihilates QR. Since Q is an elementary matrix, it follows that G_1G_1' annihilates R, as in (47). Let G_2 designate a centroid and consequently a principal axis factor matrix that generates the idempotent A. G_2 must consist of m columns from some orthogonal matrix; in fact $\|G_1G_2\|$ must be an orthogonal matrix of order n. Then since $G_1'G_2 = O$, similar principles lead to an expression for G_2 written in terms of the partitions of QR. Thus the problem of factoring a singular matrix, R, may be identified abstractly with the problem of solving linear homogeneous equations.

Kelley's rotation procedure for securing the principal components[17] also may be discussed in this context, since his procedure is designed to develop through a series of products of orthogonal matrices an n by n orthogonal matrix, the columns of which are the direction cosines of the n mutually orthogonal principal components. Although Kelley apparently intended his method to be used with non-singular matrices of data, it can readily be verified that when applied to singular matrices this method yields an orthogonal matrix that consists of the n - m columns of G_1 plus the m columns of G_2. Post-multiplying the singular R by this orthogonal matrix then yields n - m zero columns plus m non-zero columns that in effect constitute oblique (but not necessarily unit-length) factors of R. It is interesting to note that these non-zero columns also constitute orthogonal factors of R^2. Another aspect of the generality of the orthogonal matrix developed by Kelley's procedure may be of interest. If R is singular and a member of a multiplicative group that includes the idempotent A, the orthogonal matrix $\|G_1G_2\|$ is a solution in this sense for every member of the group. Also, there will exist what will be called a complementary group of rank n - m for which this same orthogonal matrix gives a solution. For this complementary group (I - A) is the multiplication unit, whereas A annihilates the group.

One other observation may be made. Let R be a sum of two matrices, R_0 designating a matrix of correlations with zero diagonal entries, and D a diagonal matrix. Then (47) becomes

$$R_0(I - A) = -D(I - A), \tag{53}$$

or

$$R_0G_1 = -DG_1, \tag{54}$$

that is, the rows of the product $R_0(I - A)$ are proportional to the rows of (I - A) when (47) holds. Further, if, with R singular, such a proportionality holds for any idempotent A, then R must equal RA. Thus the problem of factoring R may also be formulated abstractly as a problem of choosing an idempotent A for any given set of correlations, R_0, in such a way that this proportionality holds. As noted above, such a solution is a solution for an entire multiplicative group. One of the interesting questions is that of under what conditions two or more actual sets of data belonging to the same multiplicative group may arise.

16. See W. L. Ferrar, Algebra (London: Oxford University Press, 1941), pp. 100-105. MacDuffee's discussion of annihilators also is pertinent, op. cit., pp. 107-111.

17. Truman L. Kelley, Essential Traits of Mental Life, (Cambridge: Harvard University Press, 1935), pp. 1-27.

THE INDUCTIVE-DEDUCTIVE COMPARED TO THE DEDUCTIVE-DESCRIPTIVE APPROACH TO LABORATORY INSTRUCTION IN HIGH SCHOOL CHEMISTRY*

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The objectives of science education today include, in addition to the acquisition of the fundamental facts and principles of science, the ability to use the methods of science in the solution of problems and the inculcation of scientific behavior. The investigation reported here assumes all these aims of science to be important and coordinate. The primary concern was that of evaluating ways of achieving these objectives.[1] Specifically, the purpose of this tudy was to compare through experimental evaluation the learning of students instructed in such a manner as to stress the inductive-deductive approach with the learning of students instructed through the use of the more commonly found deductive-descriptive approach in the laboratory as it contributes to the objectives of high school chemistry.

Comparisons were made with respect to the relative attainment of each of four specific objectives and the total achievement involving all of these four:

1. Knowledge of basic facts and principles chemistry
2. Application of the principles of chemistry to new situations
3. Knowledge of and ability to use the scientific method with an accompanying scientific attitude
4. Ability to perform in the laboratory with resourcefulness using sound techniques

The experiment was carried out using two classes selected at random from the 1948-1949 chemistry enrollment at the University of Minnesota High School. Included as a basis for estimating the extent of the generalizability of the experimental evidence educed in the University High School experiment were seven classes selected by random techniques from the schools of Minnesota having approximately the same enrollment as University High School. Each class was randomly assigned to take part in the evaluations involving some one of the objectives.

The outside classes were taught in the manner their respective teachers commonly used. The materials taught in courses and the techniques used in the laboratory phases were essentially those used in the University High School deductive-descriptive class. This was ascertained by class visitation, teacher interview, and analysis of the textbook and laboratory manual used in each class.

Teaching in the class following the inductive-deductive approach differed from that using the deductive-descriptive approach essentially in the laboratory phases. In the inductive-deductive teaching, the laboratory was used to obtain data to be used in the solution of a problem the students had a real desire to solve. The problem was used to introduce a general principle of chemistry. The mode of attack was pupil planned with teacher guidance. The data were used to draw conclusions which were in the form of a general principle whenever possible. Teaching progressed from the particular to the general. The pupils were encouraged to recognize the value of controlled experiments, the assumptions which were basic to the laboratory plans, the desirability of clear and easy to use records, the necessity for using good laboratory techniques and careful observations, and the limitations of the data collected for drawing conclusions. In the class periods following the laboratory work the generalization was applied to numerous other related problems.

In the deductive-descriptive teaching, the laboratory exercises were taken from a representative published laboratory manual. The experi-

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* Based on a thesis submitted to the Graduate School of the University of Minnesota in partial fulfillment of the requirements for the degree of Doctor of Philosophy, 1950.

1. It is not possible to include here the comprehensive review of the literature to be found in the original study.

ments were carried out after the general principles involved had been thoroughly discussed in recitation periods. The exercises were used to describe the principles or illustrate their applications. No provisions were made for student planning of experiments; there was little call upon the student to develop and use generalizing ability. Little or no opportunity was provided for the solution of real problems under laboratory conditions. The experiments and the manner in which they were taught were of the usual school pattern as reported by Anderson.[2]

The intelligence quotient of each student was determined through the use of the Terman-McNemar Test of Mental Ability. Initial status with respect to the non-laboratory objectives was determined by administration of the achievement examinations as pretests at the beginning of the study. The achievement scores were obtained at the end of the nine-month school term by using the examinations as retests to measure the attainment of factual materials and principles, the ability to apply principles, and the ability to use the scientific method with an accompanying scientific attitude. Retention scores were obtained for those students who had not been graduated from University High School four months after the end of the school term.

The laboratory examinations were used with the University High School classes. They included a performance test for evaluating the technical skill of the students in handling apparatus, a test in which the students criticized commonly found poor techniques, and a resourcefulness test.

All achievement examinations were constructed by the writer. Each item in the examinations concerned with knowledge of facts and principles and application of principles was associated with some one of the forty-three basic chemical principles used to orient the course materials. These principles were adapted from the listing by Wise.[3] They were tried out on classes from a Minneapolis high school and two suburban schools. They were found to have reliability coefficients which were statistically significant and satisfactory for group analysis, the median being about .80. The questions although having a wide range of difficulty had an average difficulty of about 50 percent. The greatest number of them had difficulties between 25 and 75 percent.

The basic statistical techniques used in the analysis of the results were the analysis of variance and covariance. Adjustments on the criterion were made for whatever inequalities in mental ability and initial status existed among the classes by the technique of analysis of covariance. Significance levels were selected at the time of the design of the investigation for determining the acceptance or rejection of the several null hypotheses under examination. The 5 percent level was selected for rejection of null hypotheses concerning comparisons between University High School classes. The 1 percent level was selected for null hypotheses concerning the average achievement of University High School and outside classes.

In this paper the presentation of a detailed analysis of the data obtained will be restricted to the findings for only one of the objectives: knowledge of and ability to use the scientific method with an accompanying scientific attitude.

Table I presents the frequency distribution of the scores of the University High School students who took the scientific method and attitude examination. The mean achievement score of the inductive-deductive class was 38.5; the standard deviation was 5.49. The deductive-descriptive class had an average achievement score of 34.0 and a standard deviation of 6.07.

Figure 1 illustrates the use of a simple graphical method for testing the normality of this distribution of scores. In this method the cumulative percentage of frequencies of each score as the abscissa was plotted against the score value as the ordinate on probability paper. The distribution was considered normal since the plotted points lay approximately on a straight line.[4] This overcomes the inappropriateness of the usual techniques when applied to small numbers of cases.

Table II presents the basic sums of squares and cross-products for the analysis of variance and covariance.

Tables III and IV present the data and results used in testing the assumptions basic to carrying out an analysis of variance and covariance, namely, homogeneity of variance within groups and homogeneity of regression coefficients within groups.

Table V presents the final analysis of variance and covariance, from which it may be observed

2. Kenneth E. Anderson. "Summary of the Relative Achievements of the Objectives of Secondary School Science in a Representative Sampling of Fifty-Six Minnesota Schools," Science Education, XXXIII (1949), pp. 323-329.

3. Harold E. Wise. The Major Principles of Physics, Chemistry and Geology of Importance for General Education, Selected Science Services (Washington, D.C.: U. S. Office of Education, 1948), 18 pp.

4. Palmer O. Johnson. Statistical Methods in Research (New York: Prentice-Hall, Inc., 1949), pp. 160-161.

TABLE I

DISTRIBUTION OF SCORES ON SCIENTIFIC METHOD AND ATTITUDE EXAMINATION FOR INDUCTIVE-DEDUCTIVE AND DEDUCTIVE-DESCRIPTIVE CLASSES FROM UNIVERSITY HIGH SCHOOL

Score	Frequency	Cumulative Percentage	Score	Frequency	Cumulative Percentage
47	1	100	34	4	36.2
46	1	97.9	33	2	27.7
45	1	95.7	32	1	23.4
44	2	93.6	31	2	21.3
43	2	89.4	29	1	17.0
42	1	85.1	28	2	14.9
41	4	82.9	27	1	10.6
40	5	74.5	26	1	8.5
39	3	63.8	25	1	6.4
38	5	57.4	23	1	4.2
36	3	46.8	22	1	2.1
35	2	40.4			

that a significant difference (in favor of the inductive-deductive approach) was found and the null hypothesis was rejected.

The other outcome for which a significant difference in average achievement was found in favor of the inductive-deductive class was the ability to identify proper laboratory techniques when they were encountered in actual working conditions. The observed differences were in favor of the inductive-deductive approach for the objectives of knowledge of facts and principles, application of principles in new situations, performance of laboratory techniques, laboratory resourcefulness, and general achievement.

The findings from the retention examinations covering applications of principles and scientific method and attitude showed differences in mean scores in favor of the inductive-deductive class of about the same magnitude as those from the achievement examinations or retests.

The analysis of data resulting from the addition of outside deductive-descriptive classes indicated a significant advantage in favor of the University High School inductive-deductive class over one of the outside classes for both the knowledge of facts and principles and application of principles objectives. The inductive-deductive class was superior to all outside classes for the scientific method and attitude objective.

The analysis of the results obtained was carried out in the following manner.[5] An analysis of variance and covariance was carried out to test the hypothesis of homogeneity of the means of the scientific method and attitude examination scores of all the classes taking this examination. Differences were found to be significant and the hypothesis was rejected. The problem then became one of testing the significance of the differences between means of selected samples. The t-test of the significance of the differences between the means of achievement scores adjusted for differences in mental ability and pretest scores was applied. Wishart[6] showed that this was the appropriate test to use and worked out the procedures for carrying it out.

Table VI presents the results of the application of this technique to the data from the scientific method and attitude examination.

Conclusions and Recommendations

Because of the fact that the inductive-deductive class did as well or better than the deductive-descriptive class in the attainment of the general outcomes of a high school course in chemistry but was significantly superior with respect to the crucial problem of attaining knowledge of and ability in the use of the methods of science with

5. It is not possible to present comprehensive data and analyses here, but the interested reader may find them in their entirety in the original thesis on file in the Library, University of Minnesota.

6. J. Wishart. "Tests of Significance in Analysis of Covariance," Journal of the Royal Statistical Society Supplement, III (1936), pp. 79-82.

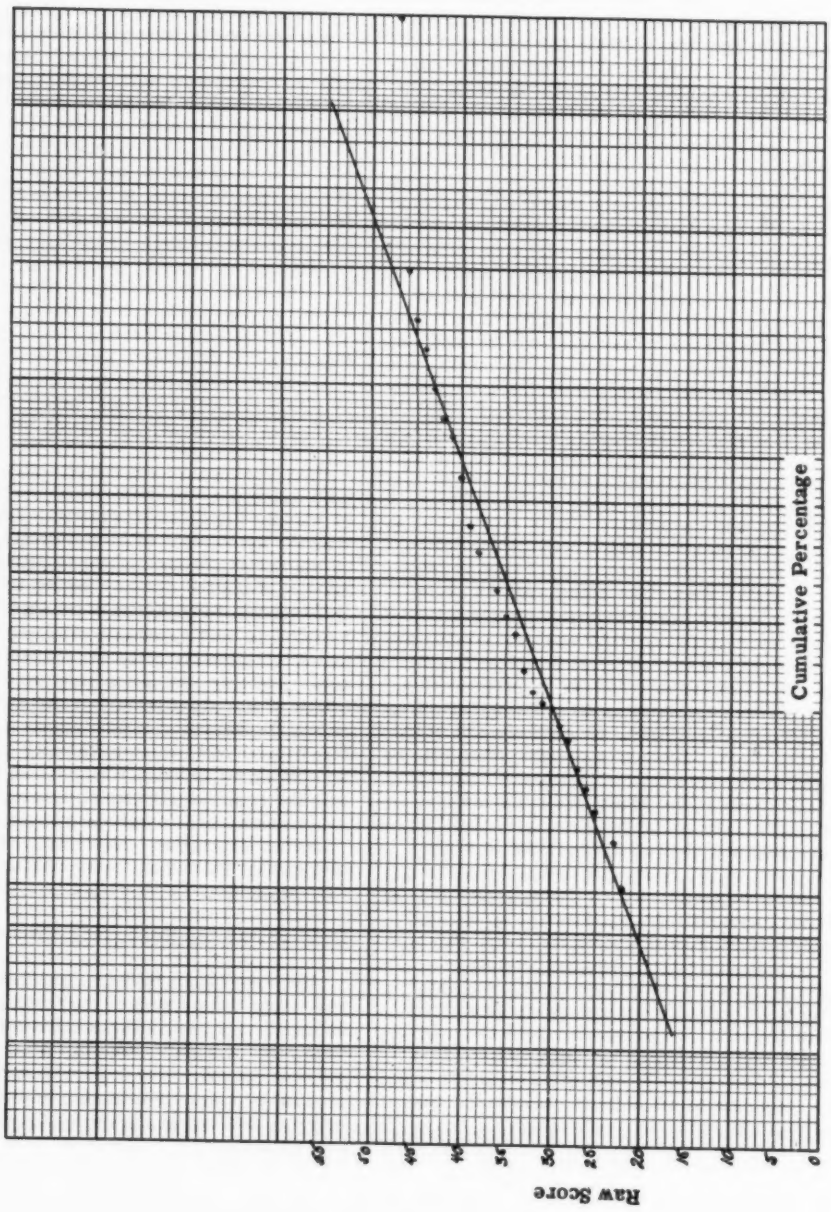

Figure 1

Normal Percentile Diagram: Cumulative percentage of raw scores on scientific method and attitude examination plotted against raw score units

TABLE II

BASIC DATA FOR ANALYSIS OF VARIANCE AND COVARIANCE OF THE SCIENTIFIC METHOD AND ATTITUDE SCORES OF THE INDUCTIVE-DEDUCTIVE AND DEDUCTIVE-DESCRIPTIVE CLASSES WITH MENTAL ABILITY AND PRETEST SCORES HELD CONSTANT

y - Total Achievement Score (deviation scores) x - Intelligence Quotient z - Pretest Scores

	N	Sum y^2	Sum x^2	Sum z^2	Sum yx	Sum yz	Sum xz
Inductive-Deductive	24	677.9583	5062.6250	850.0000	470.3750	386.5000	900.5000
Deductive-Descriptive	23	812.0000	4587.2174	815.3044	668.0000	314.0000	868.6086
		1489.9583	9649.8424	1665.3044	1138.3750	700.5000	1769.1087
Total	47	1732.2128	10189.2766	1728.8511	1499.8723	824.5745	1954.2553

TABLE III

F-TEST OF HYPOTHESIS OF HOMOGENEITY OF VARIANCES WITHIN GROUPS

	D. F.	Sum y^2	Mean Square	F	
Inductive-deductive	23	677.9583	29.4764		Hypothesis
Deductive-descriptive	22	812.0000	36.9090	1.25	accepted

TABLE IV

F-TEST OF HYPOTHESIS OF HOMOGENEITY OF REGRESSION COEFFICIENTS WITHIN GROUPS

						Adjusted Analysis	
	D. F.	Sum y^2	$b_1\Sigma yx$	$b_2\Sigma yz$	Sum y^2	Mean Square	F
Inductive-Deductive	21	677.9583	6.9735	169.6733	501.3115	23.8720	
Deductive-Descriptive	20	812.0000	60.8327	90.4669	660.7003	33.0350	1.38
						Hypothesis accepted	

TABLE V

ANALYSIS OF VARIANCE AND COVARIANCE OF SCIENTIFIC METHOD AND ATTITUDE SCORES OF I-D AND D-D CLASSES WITH MENTAL ABILITY AND PRETEST SCORES HELD CONSTANT

Source of Variation	Original Analysis D. F.	Sum y^2	$b_2\Sigma yx$	$b_2\Sigma yz$	Adjusted Analysis D. F.	Sum Squares	Mean Square	F
Within Classes	45	1489.9583	57.7520	256.9081	43	1175.2982	27.3325	
Between Classes	1	242.2545			1	123.2361	123.2361	4.51
Total	46	1732.2128	106.7155	326.9630	44	1298.5343	Hypothesis rejected	

TABLE VI

SIGNIFICANCE OF THE DIFFERENCES BETWEEN ADJUSTED MEANS OF ACHIEVEMENT SCORES ON SCIENTIFIC METHOD AND ATTITUDE EXAMINATION FOR ALL CLASSES

Comparisons	N	I. Q.	Diff. between means Pretest	Retest	Adjusted Retest	S	t	Probability	Hypothesis
Ind. -Ded.	24								
Outside E	19	23.6	3.6	12.4	7.0	1.74	4.023**	.001	reject
Ind. -Ded.	24								
Outside F	25	18.8	1.8	9.6	5.5	1.58	3.481**	.002	reject
Ind. -Ded.	24								
Outside G	30	18.0	6.8	12.3	7.4	1.62	4.568**	.001	reject
Ded. -Des.	23								
Outside E	19	16.8	1.3	7.9	4.3	1.67	2.575*	.03	reject
Ded. -Des.	23								
Outside F	25	12.0	-.5	5.1	2.8	1.24	2.258*	.04	reject
Ded. -Des.	23								
Outside G	30	11.2	4.5	7.8	4.7	1.47	3.197**	.004	reject

** Significant at the 1 percent level
* Significant at the 5 percent level
Positive differences are in favor of University High School classes.

an accompanying scientific attitude, the acceptance of the inductive-deductive method for use with future classes in chemistry at University High School was justified and this method of instruction is now in operation.

It was significantly demonstrated that this recommendation would likely be valid for schools of the state. However, before such a recommendation could be completely validated it would be necessary to carry through a cooperative experiment with a representative sampling of the schools of the state.

The author interprets the conducting of research of this type to be a valuable function of University High School. Here, in addition to providing the most efficient instruction which can be evolved for its students, the school serves in a unique way in that many future teachers of the state observe and work with its classes as part of their professional training. In this way and through the method of holding an annual conference for the high school science teachers of the state, the school has come to be looked upon as a model in which the best practices are educed through critical trial and evaluation.

As teachers of science we continuously endorse, as we should, and attempt to develop an understanding and appreciation of the scientific method on the part of the students we teach in chemistry, physics, biology, and other courses in science. To be consistent, we should make use of this method and reflect its attitudes in the study of our problems in the teaching of science. We should also behave as scientists so prospective teachers of science may see us as the prototype of the kind of teachers we expect them to become.

A COMPARISON BETWEEN THE MATCHED GROUPS AND THE COVARIANCE TECHNIQUES

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The relations between the covariance technique and the matched group formula for the standard error of a difference are illustrated by application to test data pertaining to three samples—two groups matched on the basis of ACE percentile ranks and the third group non-equivalent in this respect to the other two.

In 1941, the author reported derivations showing the relations between the analysis of variance and covariance techniques to the conventional formulas for the standard error of a difference, including the matched group formula.[1] In that paper, in addition to algebraic proofs, comparisons were made between the techniques using fictitious data. More recently the author described the use of the matched groups formula in relation to the use of the covariance techniques.[2] In his opinion this approach to the study of the covariance technique has much to recommend it. The present paper reports an application of both techniques to real data. Comparison is made between matched groups and non-equivalent groups.

In the Chicago City Junior College students are required to take a year's work in general biology entitled Biological Science 101-102. In recent years the same instruction has also been offered with laboratory in addition to lectures and discussion group sessions, a sequence of courses entitled Biological Science 111-112. Achievement at the end of both sequences of courses has been measured by means of the same three-hour comprehensive examination containing a variety of objective exercises. In June 1950, 721 students of Biological Science 101-102 and 145 students of Biological Science 111-112 took this examination at the Wright Branch of the City Junior College. Data in the form of national percentile ranks on the American Council Psychological Examination were available for 127 of the Biological Science 111-112 students and for most of the Biological Science 101-102 students. From the data pertaining to the 101-102 students two samples of 127 students each were drawn. One of these sameples was a group matched on the basis of the ACE percentile ranks with the 127 Biological Science 111-112 students. The other sample was selected at random. In Table I are reported the ACE and comprehensive examination means and standard deviations of the three samples.

The assumptions of homogeneity of variance and of homogeneity of regression were tested by the simpler methods described by the writer in 1946.[3] Both of these assumptions were satisfied.

Below are listed the results of the initial computations necessary for application of the covariance technique to the data pertaining to the Biological Science 111-112 sample and the non-equivalent or random sample of Biological Science 101-102. X_1 and Y_1 refer to the ACE and comprehensive examination data of the former and X_2 and Y_2 refers to the corresponding data of the latter. Where the subscript is dropped, reference is to both groups in combination.

$$\Sigma X_1 + \Sigma X_2 = 6420 + 5574 = 11,994 = \Sigma X$$
$$\Sigma Y_1 + \Sigma Y_2 = 23,196 + 19,772 = 42,968 = \Sigma Y$$
$$\Sigma X_1^2 + \Sigma X_2^2 = 411,052 + 324,678 = 735,730 = \Sigma X^2$$
$$\Sigma Y_1^2 + \Sigma Y_2^2 = 4,394,356 + 3,218,144 = 7,612,500 = \Sigma Y^2$$
$$\Sigma X_1 Y_1 + \Sigma X_2 Y_2 = 1,220,204 + 912,449 = 2,132,653 = \Sigma XY$$

The next step is the calculation of the sums of squares and products for total, for between groups, and for within groups in terms of deviations from the means of the scores.

1. Max D. Engelhart. "The Analysis of Variance and Covariance Techniques in Relation to the Conventional Formulas for the Standard Error of a Difference," Psychometrika, VI (August 1941), pp. 221-233.

2. Engelhart, Max D. "Suggestions With Respect to Experimentation Under School Conditions," Journal of Experimental Education, XIV (March 1946), pp. 225-244.

Σx^2, Σy^2, and Σxy for total

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N} = 735{,}730 - \frac{(11{,}994)^2}{254} = 169{,}367.68$$

$$\Sigma y^2 = \Sigma Y^2 - \frac{(\Sigma Y)^2}{N} = 7{,}612{,}500 - \frac{(42{,}968)^2}{254} = 343{,}803.24$$

$$\Sigma xy = \Sigma XY - \frac{\Sigma X \Sigma Y}{N} = 2{,}132{,}653 - \frac{11{,}994 \cdot 42{,}698}{254} = 103{,}683.79$$

Σx^2, Σy^2, and Σxy for between groups

$$\Sigma x^2 = \frac{(\Sigma X_1)^2 + (\Sigma X_2)^2}{N_1} - \frac{(\Sigma X)^2}{N} = \frac{(6420)^2 + (5574)^2}{127} - \frac{(11{,}994)^2}{254} = 2817.80$$

$$\Sigma y^2 = \frac{(\Sigma Y_1)^2 + (\Sigma Y_2)^2}{N_1} - \frac{(\Sigma Y)^2}{N} = \frac{(23{,}196)^2 + (19{,}772)^2}{127} - \frac{(42{,}968)^2}{254} = 46{,}156.78$$

$$\Sigma xy = \frac{\Sigma X_1 \Sigma Y_1 + \Sigma X_2 \Sigma Y_2}{N_1} - \frac{\Sigma X \Sigma Y}{N} = \frac{6420 \cdot 23{,}196 + 5574 \cdot 19{,}772}{127} - \frac{11{,}994 \cdot 42{,}968}{254} = 11{,}404.29$$

Σx^2, Σy^2, and Σxy for within groups

$\Sigma x^2 = 169{,}367.68 - 2{,}817.80 = 166{,}549.88$

$\Sigma y^2 = 343{,}803.24 - 46{,}156.78 = 297{,}646.46$

$\Sigma xy = 103{,}683.79 - 11{,}404.29 = 92{,}279.50$

The adjusted sum of squares of the final scores for total

$$\Sigma y^2 - \frac{(\Sigma xy)^2}{\Sigma x^2} = 343{,}803.24 - \frac{(103{,}683.79)^2}{169{,}367.68} = 280{,}329.93$$

The adjusted sum of squares of the final scores for within groups

$$\Sigma y^2 - \frac{(\Sigma xy)^2}{\Sigma x^2} = 297{,}646.46 - \frac{(92{,}279.50)^2}{166{,}549.88} = 246{,}517.60$$

The reduced sum of squares of the final scores for between groups

$280{,}329.93 - 246{,}517.60 = 33{,}812.33$

TABLE I

ACE AND COMPREHENSIVE EXAMINATION

Sample	N	ACE		Comp. Examination	
		Mean	Sigma	Mean	Sigma
Biological Science 111-112	127	50.55	26.10	182.65	35.24
Biological Science 101-102 (Random or Non-Equivalent)	127	43.89	25.10	155.69	33.19
Biological Science 101-102 (Matched)	127	50.50	25.92	158.15	33.68

In Table II is given the summarization of data usually reported in a covariance analysis.

From data in the second line of Table II one can calculate the within groups correlation coefficient between the ACE and comprehensive examination scores and the regression coefficient used in adjusting the final examination means for the non-equivalence of the samples with respect to the ACE measures.

$$r_{xy} = \frac{92,279.50}{\sqrt{166,549.88 \cdot 297,646.46}} = .4145$$

$$b_{xy} = \frac{92,279.50}{166,549.88} = .5541$$

The difference between each of the initial means and the general mean of the ACE data for both groups is 3.33. Multiplication of this value by the regression coefficient .5541 yields 1.84 to be added to My_1 and subtracted from My_2 giving the adjusted means 180.80 and 157.53. Their difference is 23.27. While the F previously given tests the significance of this difference, it can also be tested by the formula:

$$t = \frac{M'_{y_1} - M'_{y_2}}{\sqrt{\left[\frac{2}{n} + \frac{(M_{x_1} - M_{x_2})^2}{\Sigma x^2}\right] s^2}} =$$

$$\frac{180.80 - 157.53}{\sqrt{\left[\frac{2}{127} + \frac{(50.55 - 43.89)^2}{166,549.88}\right] 982.14}} = 5.87$$

A t of this size is also significant beyond the .001 level.

Where the primes refer to the adjusted comprehensive examination means, n refers to the number of students in each group, M_{x_1} and M_{x_2} are the ACE means, Σx^2 is the within groups ACE sum of squares, and s^2 is the adjusted within groups variance. It may be observed that t^2 is equal to 34.46 and approximates the value reported for F.

The results of the initial computations necessary for the application of the matched groups formula to the data for the matched samples of Biological Science 111-112 and Biological Science 101-102 students are shown in Figure One. The X's and Y's refer to gross scores and the x's and y's to the same scores expressed as deviates from the means of the groups. The subscript 1 refers to the 111-112 data, while the subscript 3 refers to the matched sample of 101-102 data.

The value for Σx_1^2 was obtained by use of the expression $\Sigma X_1^2 - \frac{(\Sigma X_1)^2}{N_1}$ and the value for $\Sigma x_1 y_1$ by use of the expression $\Sigma X_1 Y_1 - \frac{\Sigma X_1 Y_1}{N_1}$. The values for Σy_1^2, Σx_3^2, Σy_3^2, and $\Sigma x_3 y_3$ were obtained similarly. The within groups correlation between the initial and final scores was computed as follows

$$r_{xy} = \frac{89,660.48}{\sqrt{171,913.17 \cdot 301,793.55}} = .393633$$

The following formula (see page 259) was then employed in calculating t. The denominator of the formula is essentially the Lindquist-Wilks formula for the standard error of the difference between the means of matched groups.

TABLE II

ANALYSIS OF COVARIANCE AND TEST OF SIGNIFICANCE FOR THE NON-EQUIVALENT GROUPS

	Sum of Squares and Products			Adjusted or Reduced Sum of Squares	Degrees of Freedom	Adjusted or Reduced Variance
	Σx^2	Σxy	Σy^2			
Between Groups	2,817.80	11,404.29	46,156.78	33,812.33	1	33,812.33
Within Groups	166,549.88	92,279.50	297,646.46	246,517.60	251	982.14
Total	169,367.68	103,683.79	343,803.24			

$$F = \frac{33,812.33}{982.14} = 34.43$$ (Significant beyond the .001 level)

FIGURE ONE

$\Sigma X_1 = 6420$

$\Sigma X_1^2 = 411,052$

$\Sigma Y_1 = 23,196$

$\Sigma Y_1^2 = 4,394,356$

$\Sigma X_1 Y_1 = 1,220,204$

$\Sigma x_1^2 = 86,513.42$

$\Sigma y_1^2 = 157,707.27$

$\Sigma x_1 y_1 = 47,618.86$

$\Sigma X_2 = 6413$

$\Sigma X_2^2 = 409,231$

$\Sigma Y_2 = 20,085$

$\Sigma Y_2^2 = 3,320,521$

$\Sigma X_2 Y_2 = 1,056,255$

$\Sigma x_2^2 = 85,399.75$

$\Sigma y_2^2 = 144,086.28$

$\Sigma x_2 y_2 = 42,041.62$

$$\Sigma x_1^2 + \Sigma x_2^2 = \Sigma x^2 = 171,913.17$$

$$\Sigma y_1^2 + \Sigma y_2^2 = \Sigma y^2 = 301,793.55$$

$$\Sigma x_1 y_1 + \Sigma x_2 y_2 = \Sigma xy = 89,660.48$$

TABLE III

ANALYSIS OF COVARIANCE AND TEST OF SIGNIFICANCE FOR THE MATCHED GROUPS

	Sum of Squares and Products			Adjusted or Reduced Sum of Squares	Degrees of Freedom	Adjusted or Reduced Variance
	Σx^2	Σxy	Σy^2			
Between Groups	.21	85.73	38,103.63	38,014.22	1	38,014.22
Within Groups	171,913.17	89,660.48	301,793.55	255,031.56	251	1,016.06
Total	171,913.38	89,746.21	339,897.18	293,045.78		

$$F = \frac{38,014.22}{1,016.06} = 37.41$$

$$t = \frac{M_{y_1} - M_{y_2}}{\sqrt{\left(\frac{\Sigma y^2}{N_1 + N_2 - 3}\right)(1 - r_{xy}^2)\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}}$$

$$= \frac{182.65 - 158.15}{\sqrt{\left(\frac{301{,}793.55}{127 + 127 - 3}\right)(1 - .3936^2)\left(\frac{1}{127} + \frac{1}{127}\right)}}$$

$$= \frac{24.50}{3.99} = 6.14 \text{ (Significant beyond the .001 level)}$$

It may be observed that t^2, 37.70, does not differ greatly from the F of 34.43 obtained for the non-equivalent groups. It is also interesting to note the adjusted comprehensive examination mean of the non-equivalent sample of Biological Science 101-102 students 157.53 rounds off to the same whole number as the actual final mean of the matched sample of 101-102 students. Insofar as these data are worthy of generalization, it seems immaterial whether one uses matched groups in an experiment or employs the covariance technique with groups which are non-equivalent. In experimentation conducted under school conditions it usually is much more feasible to use the latter.

In further clarifying the relationships between the two procedures, it is of interest to report the results of application of the covariance techniques to the matched groups data.

Note that the Σx^2, Σxy, and Σy^2 given in the within groups row of Table III agree with the values used in calculating r_{xy} and t in the application of the matched groups formula. Also note that the adjusted sum of squares for within groups can be obtained by multiplying the unadjusted sum of squares by the expression $(1 - r_{xy})^2$; in other words, $301,793.55\ (1 - .393633)^2$ is equal to 255,031.56. The value 37.41 for F is very close to $t^2 = 37.70$ obtained through use of matched groups formula. When the final means of the matched groups were adjusted in terms of the slight difference between the initial means and this difference divided by the standard error of the difference computed by the matched groups formula, t became 6.13 and its square 37.52. Finally, when the standard error of the difference and t were computed by the formula

$$t = \frac{M'_{y_1} - M'_{y_2}}{\sqrt{\left[\frac{2}{n} + \frac{(M_{y_1} - M_{x_2})^2}{\Sigma x^2{}'}\right] s^2}}$$

the standard error was 4.00 and t = 6.12, closely approximating the 3.99 and 6.14 obtained with the matched group formula.

ACHIEVEMENT OF TWINS IN SCIENCE

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The Problem

Numerous studies have been made regarding the resemblance of twins in physical and mental characteristics, but few studies have been concerned with the resemblance of twins in academic achievement in specific areas. As part of a comprehensive study of science instruction in fifty-six Minnesota high schools,[1] the writer was able to identify several pairs of twins taking high school biology and chemistry. The problem then became one of establishing the resemblance of these twins in the various aspects of biology and chemistry instruction as measured by the examinations used.

Selection of the Sample

The biology students and chemistry students in the study had been asked to indicate their age, date of birth, sex, and grade in school as part of the testing program. Papers with the same name, age, grade in school, and date of birth, were separated from the 1980 papers and the 1352 papers in biology and chemistry. Since not all of the students so identified had provided all the information requested, it was necessary to send out letters to the schools for the twenty-one twin suspects in biology and the eighteen twin suspects in chemistry. Verification was definitely established in the case of eight like-sex twins and three pairs of unlike-sex twins in biology. Verification was definitely established in the case of seven pairs of like-sex twins and two pairs of unlike-sex twins in chemistry. The classification of the twins into fraternal or identical categories was impossible in terms of the information received.

In order to establish more adequately the twin resemblance in achievement in biology and chemistry, control groups of unrelated pairs of individuals were selected at random from the total group. Thus the control group for biology had eleven pairs of unrelated individuals which were selected from the same classes in which the twins were enrolled. The unrelated pairs in no case had the same name, thus ruling out the possibility that the pairs selected were siblings.

Achievement Factors Considered

In addition to Otis I. Q. 's the following test data were available: (1) total scores on the biology and chemistry final examinations; and (2) part scores on the final examinations. The part scores represented achievement in (a) factual information, (b) biological or chemical principles, (c) elements of the scientific method in biological or chemical situations, and (d) scientific attitudes.

The validity of the examinations was established by correlating the total scores against such criteria as teachers' marks and the scores on established tests in biology and chemistry. The reliabilities of the parts and of the total examinations were determined by using Hoyt's analysis of variance method. The reliabilities of the parts, although not high in all instances, were significant at the 1 percent level, indicating a correlation greater than zero. The correlation for the whole test was .92 in biology and .91 in chemistry. Thus, the scores used in this analysis may be considered to have been obtained by the use of valid and reliable examinations.

Statistical Analysis

For data of this kind, Johnson[2] recommends the method of analysis of variance by which one may obtain the intraclass correlation (r'). It is possible to separate the variance of a sample into two parts, (a) one arising from the natural variation of individuals treated alike, and (b) the other arising from the environmental or genetic conditions existing in the subsamples. The sum of (a) and (b) is the variance of the individuals chosen at random from the population sampled. Intraclass correlation is a ratio of two of the variances stated above, namely, $\frac{b}{a+b}$. Since r'is a ratio of two variances, we are justified in calling this ratio correlation.

The technique of analysis of variance with the

1. Kenneth E. Anderson. The Relative Achievements of the Objectives of Secondary School Science in a Representative Sampling of Fifty-Six Minnesota Schools, Doctor's Dissertation, University of Minnesota, Minneapolis, Minnesota, 1949.

2. Palmer O. Johnson. Statistical Methods in Research (New York: Prentice-Hall, Inc., 1949), p. 226.

TABLE I

CALCULATION OF L_1 IN TESTING HOMOGENEITY OF VARIANCE OF TOTAL SCORES IN BIOLOGY FOR ALL TWIN GROUPS

n_t	f_t	log n_t	n_t log n_t	$\theta'_t = \sum_t (X_{it} - \bar{X}_{.t})^2$	log θ'_t	n_t log θ'_t
2	1	.30103	.60206	50.0	1.69897	
2	1	.30103	.60206	8.0	0.90309	
2	1	.30103	.60206	180.5	2.25647	
2	1	.30103	.60206	312.5	2.49485	
2	1	.30103	.60206	60.5	1.78175	
2	1	.30103	.60206	32.0	1.50515	
2	1	.30103	.60206	0.0	0.00000	
2	1	.30103	.60206	2.0	0.30103	
2	1	.30103	.60206	162.0	2.20951	
2	1	.30103	.60206	338.0	2.52891	
2	1	.30103	.60206	0.5	-0.30103	
N = 22	log N = 1.34242		6.62266	1156.0	log $\sum_t \theta'_t$ = 3.05918	30.75740
K = 11, d.E = 1			L_1 = .120		$L_{1.01}$ = .067 $L_{1.05}$ = .120	

TABLE II

ANALYSIS OF VARIANCE OF TOTAL SCORES IN BIOLOGY FOR ALL TWIN GROUPS

Source of Variation	D. F.	Sum of Squares	Mean Square	F	Hypothesis
Within Pairs	11	1146.000	104.181		P< .01
Between Pairs	10	6069.454	606.945	5.83	Reject
Total	21	7215.454			

resulting intraclass correlation was applied on the twin and control groups for six sets of scores representing: (1) the Otis I. Q., (2) the acquisition of factual information in biology or chemistry, (3) the understanding of biological or chemical principles, (4) the understanding and application of the elements of the scientific method in biological or chemical situations, (5) the acquisition of scientific attitudes, and (6) the combined achievement in biology or chemistry as represented by the total examinations.

Since homogeneity of variances as regards the trait under consideration is the fundamental assumption underlying the analysis of variance, L-tests were run for the six comparisons for the control and twin group in both biology and chemistry. All of the L values obtained indicated homogeneity of variances either at the 5 or 1 percent level. Table I shows the calculation of the L_1 test for the experimental group in biology on total test scores.

Once it has been established that the variability of the measure of the trait under consideration is the same for all twin pairs, one may proceed with the test of the hypothesis that the acquisition of an ability by an individual is independent of the particular twin pair to which the individual belongs. This is done by calculating the F ratio as shown in Table II for experimental group in biology on the total test scores. If the F thus obtained is significant at a specified level of significance, one may calculate the intraclass correlation by dividing "the mean square between pairs minus the mean square within pairs" by "the sum of the mean squares for the within and between pairs." Thus for the experimental group in biology for total test score, the intraclass correlation becomes:

$$r' = \frac{502.764}{711.126} = .707$$

A positive intraclass correlation is obtained when the mean square between classes is significantly larger than the mean square within classes. The reverse is true for a negative intraclass correlation.

Statistical Results

Tables III and IV show the essential values calculated for all six sets of data for all twin pairs, like-sex twin pairs, and control pairs for biology and chemistry.

An examination of Tables III and IV allows us to draw the following conclusions regarding intelligence:

(1) The I. Q. of an individual is not independent of the twin pair to which he belonged or the intraclass correlation between twins significantly greater than zero. The intraclass correlation for all twin pairs was .50 for biology and .61 for chemistry. When the unlike-sex twins were removed the correlations became .63 and .86 respectively.

(2) The I. Q. of an individual was independent of the unrelated pair to which he belonged or the intraclass correlation was not significantly greater than zero. The intraclass correlations for the unrelated pairs of the control group were .19 and -.31 for biology and chemistry respectively, neither correlation being significant at the 5 percent level.

An examination of Table III for biology allows us to draw the following conclusions:

(1) The achievements of an individual in biology were not independent of the twin pair to which he belonged or the intraclass correlations were significantly greater than zero at the 5 or 1 percent level for all twin pairs for all parts of the test and total test, except in the case of acquisition of factual concepts. This was not true for the control group of unrelated individuals, except in the case of acquisition of scientific attitudes, where a significant intraclass correlation of .51 at the 1 percent level was obtained.

(2) The intraclass correlations for all twin pairs were as follows: .43 for factual concepts, .81 for understanding of principles, .54 for understanding and application of the scientific method, .59 for acquisition of scientific attitudes, and .71 for total understanding and acquisition in biology.

(3) The intraclass correlations for the same abilities with unlike-sex twins removed were as follows: .65, .98, .51, .52, and .76. The correlations of .65, .98, and .76 were significant at the 5 or 1 percent level and were of greater magnitude than was the case for all twin pairs. The correlations of .51 and .52 were not significant at the 5 percent level and of less magnitude than was the case for all twin pairs.

An examination of Table IV for chemistry allows us to draw the following conclusions:

(1) The achievements of an individual in chemistry were not independent of the twin pair to which he belonged or the intraclass correlations were significantly greater than zero at the 5 or 1 percent level for all twin pairs for all parts of the test and total test except in the case of the understanding of principles, and the acquisition of scientific attitudes. This was not true for the control group of unrelated individuals where all of the intraclass correlations were not significantly greater than zero.

(2) The intraclass correlations for all twin pairs were as follows: .68 for acquisition of factual concepts, .41 for understanding of principles, .53 for understanding and application of elements of the scientific method, .51 for acquisition of scientific attitudes, and .59 for total acquisition and understanding in chemistry.

TABLE III

ESSENTIAL VALUES FOR THE TWIN AND CONTROL GROUPS IN BIOLOGY

Ability		11 Pairs All Twins	8 Pairs Like Sex	11 Pairs Control
I. Q.	L_1	.125	.128	.233
	F	3.03 *	4.45 *	1.47
	P	.05>P>.01	.05>P>.01	P>.05
	r'	.503*	.633*	.191
Facts	L_1	.206	.296	.219
	F	2.52	4.67 *	1.53
	P	P>.05	.05>P>.01	P>.05
	r'	.432	.647*	.210
Principles	L_1	.260	.276	.307
	F	9.37 **	8.99 **	1.67
	P	P<.01	P<.01	P>.05
	r'	.807**	.978**	.251
Elements of Scientific Method	L_1	.222	.233	.164
	F	3.36 *	3.05	2.68
	P	.05>P>.01	P>.05	P>.05
	r'	.542*	.507	.457
Scientific Attitudes	L_1	.412	.374	.345
	F	3.83 *	3.19	3.11 *
	P	.05>P>.01	P>.05	.05>P>.01
	r'	.586*	.523	.513*
Total Score	L_1	.119	.241	.116
	F	5.83 **	7.29 **	2.43
	P	P<.01	P<.01	P>.05
	r'	.707**	.758**	.416

* 5 percent level
** 1 percent level

TABLE IV

ESSENTIAL VALUES FOR THE TWIN AND CONTROL GROUPS IN CHEMISTRY

Ability		9 Pairs All Twins	7 Pairs Like Sex	9 Pairs Control
I. Q.	L_1	.172	.268	.132
	F	4.13 *	7.18 *	0.53
	P	.05>P>.01	.05>P>.01	P>.05
	r'	.610*	.860*	-.308
Facts	L_1	.160	.189	.254
	F	5.22 *	2.17	0.29
	P	.05>P>.01	P>.05	P>.05
	r'	.678*	.369	-.545
Principles	L_1	.255	.385	.177
	F	2.41	5.27 *	3.68
	P	P>.05	.05>P>.01	P>.05
	r'	.413	.681*	-.460
Elements of Scientific Method	L_1	.101	.135	.222
	F	3.24 *	2.27	0.47
	P	.05>P>.01	P>.05	P>.05
	r'	.528*	.387	-.358
Scientific Attitudes	L_1	642	.570	.389
	F	3.10	1.88	1.42
	P	P>.05	P>.05	P>.05
	r'	.512	.305	.175
Total Score	L_1	.225	.223	.203
	F	3.87 *	2.60	0.36
	P	.05>P>.01	P>.05	P>.05
	r'	.589*	.445	-.643

* 5 percent level
** 1 percent level

(3) The intraclass correlations for the same abilities with unlike-sex twins removed were as follows: .37, .68, .39, .31, and .45. Only one of these correlations was significant at the 5 percent level. This was the case on the understanding of principles of chemistry and represents an increase over the correlation for all twin pairs.

General Conclusions

It has been clearly demonstrated by the results of this investigation that intelligence of an individual, as represented by the Otis intelligence quotient, was not independent of the twin pair to which he belonged, and that when the unlike-sex twins were removed, an increase in the intraclass correlation resulted. This same conclusion was not warranted in the case of unrelated pairs of individuals drawn from the same classes in which the twins were enrolled. These conclusions are no different from those drawn by previous investigators regarding the influence of heredity on intelligence.

It has been demonstrated by the results of this investigation that some of the academic abilities as measured by part examinations and acquired by an individual in a year of study, were not independent of the twin pair to which he belonged. This was especially true in case of the ability to understand and apply the principles of biology and chemistry. This ability is generally recognized by most science educators to be an important objective of science instruction. The same conclusions for the abilities measured by the part examinations are not warranted for the unrelated pairs of individuals in the control groups. The removal of unlike-sex twins raised the intraclass correlations for some of the abilities and lowered the intraclass correlations for other abilities. Not all of the correlations were statistically significant.

It has been demonstrated by the results of this investigation that achievement in biology and chemistry as measured by the total examinations, and as acquired by an individual in a year of study, was not independent of the twin pair to which he belonged. This conclusion was not warranted in the case of the unrelated pairs of individuals drawn from the same classes in which the twins were enrolled. The removal of unlike-sex twins in the biology calculations raised the intraclass correlation from .71 to .76. This was not the case in chemistry where the intraclass correlation became insignificant. Previous studies have demonstrated the resemblance of twins in mental characteristics as measured by intelligence tests. This study substantiates those findings. This study was not primarily concerned with the resemblance of twins in intelligence but with the resemblance of twins in certain defined academic abilities in the fields of high school chemistry and biology. It has been demonstrated by the present study that twins exhibit a high degree of resemblance in the defined academic abilities and that this resemblance is not present in unrelated individuals. One might well raise the question whether these academic abilities are highly related to intelligence. This is true to some degree, as positive correlations ranging from .40 to .70 were obtained when the part-test scores were correlated with the intelligence test scores. However, for all twin pairs, the total test scores in biology and chemistry produced intraclass correlations of .71 and .59 as contrasted to the intraclass correlations for intelligence quotients of .50 and .61 respectively.

As far as this study is concerned, it is evident that heredity exerts an influence not only in the realm of intelligence but also in the ability to acquire knowledge of science as measured by the examinations used.

CURVE FITTING IN POLAR COORDINATES

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Curves are usually fitted to empirical data which are expressed in Cartesian coordinates. However, in certain inquiries, it may prove desirable to plot the scatter diagram in polar coordinates, and to employ statistical measures expressed in terms of angular distances and radius.

The simple Archimedes' Spiral is well adapted for representing any phenomenon in which the spatial position of a point is proportional to its angular distance. The unwinding of certain types of springs might exemplify this phenomenon.

This same curve ($R = a + b\theta$) can also be used for the expression of trends. There are perhaps many instances in which a better visual representation is obtained by employing polar coordinates.

Percentages can easily be converted into angular measure, by taking 360^o, rather than 100, as the base. Any given set of data can either be taken in terms of angles, or else transformed into angular measure. Furthermore, any data or equation expressed in Cartesian coordinates can be transformally mapped in polar coordinates, and vice-versa, by means of simple transformation equations.

In the following illustration, an Archimedes' Spiral is fitted to a scatter diagram consisting of 18 observed spatial positions (R) of a point, at various angular distances (θ) from the origin, taken at zero. These data are imaginary; and it is assumed that, in the phenomenon under investigation, the slope and scatter are such as to enable the angular distance to be determined, say, at 60^o, rather than 420^o.

The method of least-squares is applied here to determine the estimating equation. The normal equations are derived as follows:

Let R_C be a computed trend value, and let R be an observed value. In the least-squares method, $(R - R_C)^2$ must be at a minimum. Since the equation of the Archimedes' Spiral is $R_C = a + b\theta$, we obtain:

$$\Sigma(R - R_C)^2 = \Sigma\left[R - (a + b\theta)\right]^2$$
$$= \Sigma R^2 - 2a\Sigma R - 2b\Sigma R\,\theta + Na^2 + 2ab\Sigma\,\theta + b^2\Sigma\,\theta^2.$$

Denoting this expression as Φ, and taking the partial derivative with respect to a, we obtain:

$$\frac{\delta\Phi}{\delta a} = -2\Sigma R + 2Na + 2b\Sigma\theta.$$

By setting this partial derivative equal to zero, we obtain the minimum value of the curve.

$$-2\Sigma R + 2Na + 2b\Sigma\theta = 0, \quad \text{or}$$

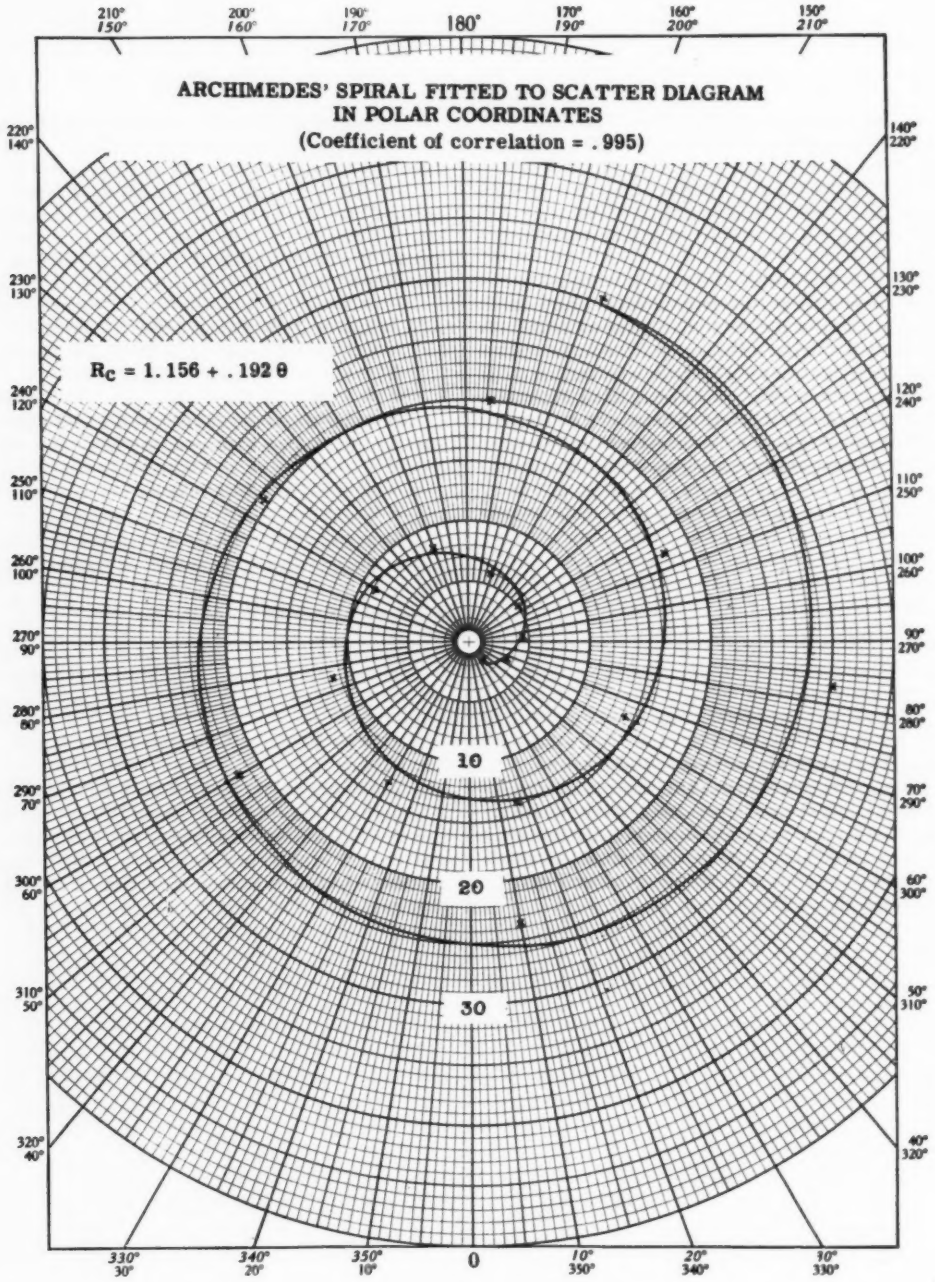
normal equation (1), which is: $\Sigma R = Na + b\Sigma\theta$.

Setting the partial derivative with respect to b equal to zero, we obtain:

$$-2\Sigma R\,\theta + 2a\Sigma\,\theta + 2b\Sigma\,\theta^2 = 0, \quad \text{or}$$

normal equation (2), which is: $\Sigma R\theta = a\Sigma\theta + b\Sigma\theta^2$.

The following are the computations for the determination of the estimating equation and the variances. In the accompanying graph, the Archimedes' Spiral is shown, as fitted to the scatter diagram.


ARCHIMEDES' SPIRAL FITTED TO SCATTER DIAGRAM
IN POLAR COORDINATES
(Coefficient of correlation = .995)
$R_C = 1.156 + .192\,\theta$
10
20
30
180°
0

FITTING OF ARCHIMEDES' SPIRAL; AND COMPUTATION OF TOTAL VARIANCE, EXPLAINED VARIANCE, AND UNEXPLAINED VARIANCE, FOR SPATIAL POSITIONS, AS ESTIMATED BY ANGULAR DISTANCE

Angular distance in degrees	Angular distance in radians (θ)	Observed spatial position in inches (R)	$\theta \times R$	θ^2	Computed position (R_C)	$(R - R_C)^2$	$(R - \bar{R})^2$	$(R_C - \bar{R})^2$
39	6.807	2.0	13.614	46.335	2.463	.214	149.109	138.016
64	11.169	3.5	39.092	124.747	3.300	.040	114.726	119.050
94	16.402	4.5	73.809	269.026	4.305	.038	94.304	98.129
126	21.987	5.0	109.935	483.428	5.378	.143	84.843	78.022
161	28.093	6.0	168.558	789.217	6.550	.303	67.421	58.691
200	34.900	8.2	286.180	1218.010	7.857	.118	36.132	40.373
241	42.055	8.9	374.290	1768.623	9.231	.110	28.207	24.800
285	49.731	11.6	576.880	2473.172	10.704	.803	6.817	12.299
330	57.584	13.3	765.867	3315.917	12.212	1.184	.830	3.996
376	65.612	13.9	912.007	4304.935	13.754	.021	.097	.209
424	73.989	14.2	1050.644	5474.372	15.362	1.350	.000	1.325
474	82.712	17.9	1480.545	6841.275	17.037	.745	13.609	7.986
534	93.184	20.0	1863.680	8683.258	19.047	.908	33.513	23.387
595	103.829	20.5	2128.495	10780.461	21.091	.349	39.552	47.334
660	115.171	22.0	2533.762	13264.359	23.269	1.610	60.669	82.047
730	127.386	23.7	3019.048	16227.193	25.614	3.663	90.041	130.028
803	140.123	30.1	4217.702	19634.455	28.060	4.162	252.460	191.795
878	153.212	30.5	4672.966	23474.917	30.573	.005	265.332	267.715
Total	1,223.946	255.8	24,287.074	119,172.700		15.766	1,337.662	1,325.202
Mean		14.211				.876	74.315	73.622

Normal equations: (1) $\Sigma R = Na + b\Sigma\theta$.
(2) $\Sigma R\theta = a\Sigma\theta + b\Sigma\theta^2$.

Substitution of Computed Values: (1) $255.8 = 18a + 1223.946b$. $a = 1.156$
(2) $24,287.074 = 1223.946a + 119,172.700b$. $b = .192$

Equation of Fitted Archimedes' Spiral: $R_C = 1.156 + .192\theta$.